The Euler Equation of Gas-Dynamics

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In this lecture we study some properties of the Euler equations of gasdynamics,

$$
\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) &= 0\,, \\
\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}\right) + \nabla p &= \rho\boldsymbol{a}\,, \\
\frac{\partial p}{\partial t} + \boldsymbol{u}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{u} &= 0\,,
\end{aligned}
\tag{1}
$$

where ρ, p and $\boldsymbol{u}$ denote that gas density, pressure and (bulk) velocity. The last equation can be recovered from the internal energy equation (see next section) assuming an ideal gas which satisfies $\rho e = p/(\gamma-1)$, where γ is the specific heat ratio.

1 The Internal Energy Equation

In order to derive the internal energy equation for an ideal gas, we start from the conservative form of the total energy density:

$$
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e\right) + \nabla\cdot\left[\left(\frac{1}{2}\rho u^2 + \rho e + p\right)\boldsymbol{u}\right] = \rho\boldsymbol{u}\cdot\boldsymbol{a}\,. \tag{2}
$$

Now consider the temporal evolution of the kinetic term. Using the momentum Eq. in (1) we obtain:

$$
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = \frac{u^2}{2}\partial_t\rho + \rho\boldsymbol{u}\cdot\partial_t\boldsymbol{u} = -\frac{u^2}{2}\nabla\cdot(\rho\boldsymbol{u}) + \rho\boldsymbol{u}\cdot\left[\boldsymbol{a} - \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \frac{\nabla p}{\rho}\right]\,.
$$

We can now replace the second term in the square bracket using the vector identity $\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \nabla(u^2/2) - \boldsymbol{u}\times(\nabla\times\boldsymbol{u})$ so that, together with the continuity equation, the previous equation reads

$$
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2\right) = -\nabla\left(\frac{\rho u^2}{2}\boldsymbol{u}\right) - \boldsymbol{u}\cdot\nabla p + \rho\boldsymbol{u}\cdot\boldsymbol{a}\,.
$$

Substituting in Eq. (2) we obtain

$$
\frac{\partial}{\partial t}(\rho e) + \nabla\cdot(\rho e\boldsymbol{u}) + p\nabla\cdot\boldsymbol{u} = 0\,. \tag{3}
$$

The last term represents the work done by compression ($\nabla\cdot\boldsymbol{u} < 0$) or expansion ($\nabla\cdot\boldsymbol{u} > 0$) of the gas.

1.1 Relation to the 1^{st} Law of Thermodynamics

Notice that Eq. (3) is actually the first law of thermodynamics, just written in a different way. To prove this equivalence, we start directly from the 1^{st} law which, in our notations, can be written as,

$$de + pdV = \delta Q = 0 \quad (\text{adiabatic})\,, \tag{4}$$

where $V = 1/\rho$ represents the volume divided by the mass and e is the specific internal energy. The previous equation holds in a volume of fluid as it moves along a streamline and, after dividing by dt, the derivatives must be understood as convective (Lagrangian) derivative ($d/dt = \partial_t + \boldsymbol{u}\cdot\nabla$):

$$\frac{\partial e}{\partial t} + \boldsymbol{u}\cdot\nabla e + p\left[\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) + \boldsymbol{u}\cdot\nabla\left(\frac{1}{\rho}\right)\right] = 0 \quad\Longrightarrow\quad \frac{\partial e}{\partial t} + \boldsymbol{u}\cdot\nabla e - \frac{p}{\rho^2}\left(\frac{\partial\rho}{\partial t} + \boldsymbol{u}\cdot\nabla\rho\right) = 0\,.$$

Now, using the continuity equation, $\partial_t = -\nabla\cdot(\rho\boldsymbol{u})$, one obtains

$$\frac{\partial e}{\partial t} + \boldsymbol{u}\cdot\nabla e + \frac{p}{\rho}\nabla\cdot\boldsymbol{u} = 0\,, \tag{5}$$

or, written using the Lagrangian derivative,

$$\frac{de}{dt} = -\frac{p}{\rho}\nabla\cdot\boldsymbol{u}\,, \tag{6}$$

which, after multiplication by ρ together with the continuity equation, gives again, Eq. (3).

A different form of the energy equation can be obtained for an ideal gas by assuming $\rho e = p/(\gamma-1)$ (thas it, the gas is adiabatic). After dividing Eq. (3) (or the 3^{rd} by ρ^γ we obtain

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) + \frac{\gamma p}{\rho^{\gamma+1}}\frac{d\rho}{dt} + \frac{\gamma p}{\rho^\gamma}\nabla\cdot\boldsymbol{u} = 0\,.$$

Using the continuity Eq. $d\rho/dt = -\rho\nabla\cdot\boldsymbol{u}$, terms simplify and one is left with

$$\frac{\partial s}{\partial t} + \boldsymbol{u}\cdot\nabla s = \frac{ds}{dt} = 0\,. \tag{7}$$

where $s = p/\rho^\gamma$. Eq. (7) simply states the conservation of entropy along a streamline as the fluid moves, reflecting the adiabatic nature of the gas.

1.2 Definition of Temperature

Temperature can be defines in a statistical sense as

$$p = nk_BT = \frac{nm\left\langle w^2\right\rangle}{3} \qquad\Longrightarrow\qquad T = \frac{m\left\langle w^2\right\rangle}{3k_B} \tag{8}$$

where k_B is the Boltzmann constant, p is the pressure, n is the gas number density. For a system in local thermodynamic equilibrium the distribution function becomes a Maxwellian,

$$f = n\left(\frac{m}{2\pi kT}\right)^{3/2}\exp\left[-\frac{mw^2}{2kT}\right] \tag{9}$$

and the definition of the temperature given by Eq. (8) can be directly verified. Indeed, using the fact that

$$\int_0^\infty x^{2n}\exp^{-x^2/a^2}dx = \sqrt{\pi}\frac{(2n)!}{n!}\left(\frac{a}{2}\right)^{2n+1}$$

we can compute the average value of $\left\langle w^2\right\rangle$ in spherical coordinates in the velocity space, where $d^3w = 4\pi w^2dw$:

$$\left\langle w^2\right\rangle = \frac{\int_0^\infty w^4\,4\pi dw}{\int_0^\infty w^2\,4\pi dw} = \frac{3nk_BT/m}{n} = \frac{3k_BT}{m}$$

2 Simple Analytical Solutions

2.1 Time-independent Solution

2.1.1 Constant Uniform Medium

Perhaps the simplest solution involves a static ($\boldsymbol{u} = 0$) uniform fluid with constant density, $\rho = \rho_0 = \text{const}$ and pressure $p = p_0 = \text{const}$ and no acceleration ($\boldsymbol{a} = 0$). It can be easily verified that this condition satisfies the system (1).

2.1.2 Hydrostatic Equilibrium

Equilibrium conditions satisfy $\partial_t = 0$ and can be either static ($\boldsymbol{u} = 0$) or stationary ($\boldsymbol{u} \neq 0$). A simple solution can be obtained by considering a static medium under the action of gravity. We consider here a constant gravitational field so that $\boldsymbol{a} = (0, 0, -g)$. Assuming that flow quantities depend only on the vertical coordinate (z) and neglecting variation in the horizontal plane ($\partial_x = \partial_y = 0$), only the equation of motion is non-trivial:

$$\frac{dp}{dz} = -\rho g \tag{10}$$

The previous ordinary differential equation is the (one-dimensional) hydrostatic balance equation. It can be solved once a relation between p and ρ has been specified.

- Constant density: for an incompressible fluid, $\rho = \rho_0 = \text{const}$ and Eq. (10) has the simple solution

  $$p(z) = p(z_0) - \rho g(z - z_0) \tag{11}$$

  The previous relation is know as Stevin's law (legge di Stevino).

- Isothermal fluid, for which $p = a^2\rho$, where a is the isothermal speed of sound. In this case, Eq. (10) can be solved giving

  $$p(z) = p(z_0)e^{-g(z-z_0)/a^2} \tag{12}$$

  Note that a^2/g is the atmospheric scale height.

2.1.3 Bernoulli's Law

We now show how, under specific conditions, the Eq. of motion (the second in Eqns 1) can be manipulated to yield Bernoulli's law. Using the identity

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = \nabla \left(\frac{1}{2}\boldsymbol{u}^2\right) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})$$

we rewrite the second equation in (1) as

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \left(\frac{1}{2}\boldsymbol{u}^2\right) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) + \frac{\nabla p}{\rho} = \boldsymbol{a}$$

Since $\nabla(p/\rho) = (\nabla p)/\rho - (p/\rho^2)\nabla\rho$ we then obtain

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla \left(\frac{1}{2}\boldsymbol{u}^2 + \frac{p}{\rho}\right) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) + \frac{p}{\rho^2}\nabla\rho = \boldsymbol{a} \tag{13}$$

If the external force is conservative, then a potential can be defined such that $\boldsymbol{a} = -\nabla\varphi$. In addition, assuming a stationary flow ($\partial_t = 0$) and the incomprimibility condition ($\rho = \text{const}$), the previous equation further simplify to

$$\nabla\left(\frac{1}{2}\boldsymbol{u}^2 + \frac{p}{\rho} + \varphi\right) = (\boldsymbol{u} \times \nabla \times \boldsymbol{u}) \tag{14}$$

This equation can be projected along a fluid streamline (the fluid direction given by $\boldsymbol{u}$), the term on the right hand side vanishes and one is left with

$$\boldsymbol{u} \cdot \nabla\left(\frac{1}{2}\boldsymbol{u}^2 + \frac{p}{\rho} + \varphi\right) = 0 \tag{15}$$

This implies that in a steady inviscid and incompressible flow in an external conservative field the quantity inside the round brakets is constant:

$$b_l = \frac{1}{2}\boldsymbol{u}^2 + \frac{p}{\rho} + \varphi \tag{16}$$

Note that b_l is, in general, different on different streamlines if the flow has non-zero vorticity ($\nabla \times \boldsymbol{u} \neq 0$). However, if the flow happens to be also irrotational ($\nabla \times \boldsymbol{u} = 0$) then b_l defined by Eq. (16) is constant *everywhere* in the flow.

2.2 Time-Dependent Solutions

Analytical solutions in the general time-dependent case can be, in general, obtained numerically. However, there are simple cases that are worth discussing and in which the hyperbolic nature of the underlying partial differential equations can be understood.

2.2.1 Uniform Advection

Consider a generic density profile $\rho(x,t)$ in a fluid with constant velocity $\boldsymbol{u} = u_0\hat{\boldsymbol{\imath}}$ and constant pressure. Then only the first of Eqns (1) is non-trivial, yielding

$$\frac{\partial\rho(x,t)}{\partial t} + u_0\frac{\partial\rho(x,t)}{\partial x} = 0 \tag{17}$$

Eq. (17) is known as the linear advection (or transport) equation and it can be considered as the proto-type of all hyperbolic partial differential equations (PDE). Hyperbolic PDE imply, as we shall see, that information propagates across domain at a finite speed.

It is easy to verify that the solution of Eq. (17) is a uniform shift of *any* initial profile. That is, given the initial condition

$$\rho(x,0) = f(x) \qquad \text{at} \quad t = 0\,,$$

then Eq. (17) admits the solution

$$\rho(x,t) = \rho(x - u_0t, 0) \equiv f(x - u_0t) \tag{18}$$

which describes a uniform (rigid) translation of the initial density profile. This can be easily verified by straightforward differentiation (setting $\xi = x - u_0t$).

A very useful concept in the theory of hyperbolic PDE is given by the notion of characteristic curve. For Eq. (17) characteristic curves are defined by the ordinary differential equation

$$\frac{dx}{dt} = u_0 \tag{19}$$

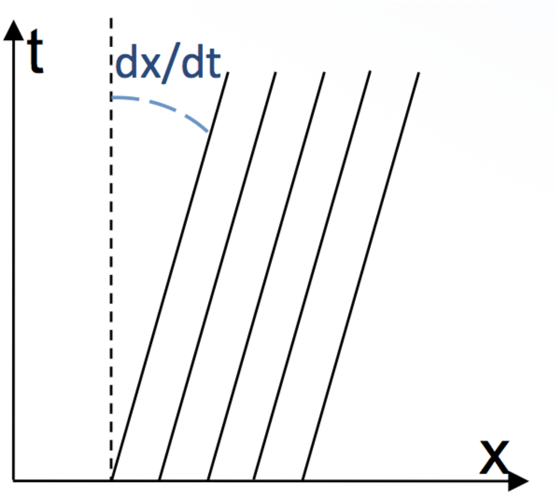

Figure 1: Characteristic curves for the linear advection equation.

and are simply parallel straight lines. Differentiating $\rho(x,t)$ with respect to time along a characteristic curve $x = x(t)$, one obtains

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{dx(t)}{dt}\frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x} = 0\,,$$

e.g., the solution is *constant* along characteristics.

At any point (x,t) we can trace the characteristic back to the initial position, see Figure 1.

2.2.2 Sound Waves: Linearizing Euler's equations

The previous (exact) solution shows how wave propagation is a natural concept when dealing with hyperbolic PDE. The same concept applies, of course, to the full system of equations (1) although simple analytical solutions can not be found due to their intrinsic nonlinearity.

For this reason, we now introduce a perturbative approach that can be used to linearize the equations of gas-dynamics. A linear approach can always be studied in terms of Fourier modes and can be very instructive on the nature of waves, at least in the limit of small perturbations. The term *perturbation* usually implies that the perturbed quantities are small compared to the unperturbed quantities and the quadratic (or higher-order) terms of these quantities can be neglected. This is precisely the meaning of the *linear perturbation technique.*

Consider a static, homegenous medium with constant density and pressure (ρ_0 and p_0). Assuming small perturbations ($|\rho_1| \ll \rho_0$, $|p_1| \ll p_0$) so that $\rho(x,t) = \rho_0 + \rho_1(x,t)$, $p(x,t) = p_0 + p(x,t)$ and $\boldsymbol{u}(x,t) = \boldsymbol{u}_1(x,t)$ we rewrite the system (1) as

$$\left\{\begin{array}{lcl} \dfrac{\partial \rho_1}{\partial t} + \rho_0 \nabla\cdot\boldsymbol{u}_1 & = & 0 \\ \rho_0 \dfrac{\partial \boldsymbol{u}_1}{\partial t} + \nabla p_1 & = & 0 \\ \dfrac{\partial p_1}{\partial t} + \gamma p_0 \nabla\cdot\boldsymbol{u}_1 & = & 0 \end{array}\right. \tag{20}$$

Note that only first-order terms have been retained while second- or higher-order terms (such as $\rho_1\boldsymbol{u}_1$) have been neglected. This assumption is justified in the regime of small perturbations.

Given the linear nature of the system we can now employ a plane wave decomposition and write a generic perturbation in the from $\propto e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ so that $\partial_t \to -i\omega$ while $\nabla \to i\boldsymbol{k}$. Since the perturbation analysis is linear, the principle of superposition holds. Eqns (20) then become

$$\left\{\begin{array}{lcl} -i\omega\rho_1 + \rho_0 i\boldsymbol{k}\cdot\boldsymbol{u}_1 & = & 0 \\ -i\omega\rho_0\boldsymbol{u}_1 + i\boldsymbol{k}p_1 & = & 0 \\ -i\omega p_1 + \gamma p_0 i\boldsymbol{k}\cdot\boldsymbol{u}_1 & = & 0 \end{array}\right. \tag{21}$$

Without loss of generality, we can assume $\boldsymbol{k} = k\hat{\boldsymbol{i}}$ in the x direction and write the system (21) as

$$\left\{\begin{array}{lcl} -\omega\rho_1 + \rho_0 k u_1 & = & 0 \\ -\omega\rho_0 u_1 + k p_1 & = & 0 \\ -\omega p_1 + \gamma p_0 k u_1 & = & 0 \end{array}\right. \tag{22}$$

where now u_1 is the x-component of the velocity perturbation. The previous system is a homogeneous linear system in ρ_1, u_1 and p_1 which, in matrix notations, can be easily cast as

$$\left(\begin{array}{ccc} -\omega/k & \rho_0 & 0 \\ 0 & -\omega/k & 1/\rho_0 \\ 0 & \gamma p_0 & -\omega/k \end{array}\right)\left(\begin{array}{c} \rho_1 \\ u_1 \\ p_1 \end{array}\right) = 0\,. \tag{23}$$

Eq. (23) has a non-trivial solution if the determinant of the 3×3 matrix vanishes:

$$-\left(\frac{\omega}{k}\right)^3 + \frac{\omega}{k}\frac{\gamma p_0}{\rho_0} = 0 \qquad \Longrightarrow \qquad \left(\frac{\omega}{k}\right) = \pm\sqrt{\frac{\gamma p_0}{\rho_0}} \tag{24}$$

which is the solution of our dispersion relation. The quantity

$$c_s \equiv \sqrt{\frac{\gamma p_0}{\rho_0}} \tag{25}$$

defines the adiabatic speed of sound and was first obtained by Newton (1689), who assumed pertrubations to be isothermal ($\gamma \to 1$). At the temperature of $T = 273$ K (0°) the adiabatic sound speed is approximately 332 m/s.

Sound waves are longitudinal waves because particles of the medium through which the sound is transported vibrate parallel to the direction of the wave. This can be easily recognized by the fact that $\boldsymbol{k}$ and $\boldsymbol{u}_1$ are parallel. The result of such longitudinal vibrations is the creation of compressions and rarefactions within the air.

3 The Vorticity Equation. Barotropic and Incompressible Flows

By using the vector identity, the momentum equation can be written as

$$\frac{\partial\boldsymbol{u}}{\partial t} + \frac{1}{2}\nabla u^2 - \boldsymbol{u}\times(\nabla\times\boldsymbol{u}) = -\frac{1}{\rho}\nabla p + \boldsymbol{a}\,.$$

Taking the curl of the previous equation and considering a conservative force we obtain

$$\frac{\partial\boldsymbol{\omega}}{\partial t} = \nabla\times(\boldsymbol{u}\times\boldsymbol{\omega}) + \frac{1}{\rho^2}\nabla\rho\times\nabla p \tag{26}$$

where $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ is the *vorticity.*

The last term on the right hand side is the baroclinic term. It describes the production of vorticity due to a misalignment between pressure and density gradients.

Barotropic Fluids. A barotropic equation of state assume that the pressure is a function of the density only, $p = p(\rho)$. In this case the last term on the right hand side of the vorticity Eq. (26) vanishes yielding

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) \tag{27}$$

In astrophysics, barotropic fluids are found while studying stellar interiors, degenerate matter (white dwarf or neutron stars) or the interstellar medium. One common class of barotropic model used in astrophysics is a polytropic fluid, for which $p \propto \rho^\gamma$. A special case of a barotropic fluid is represented by an isothermal flow, for which $p = c_s^2\rho$, where c_s is the isothermal speed of sound.

Incompressible Flows. In an incompressible flow, the material density in a fluid parcel is constant ($d\rho/dt = 0$). From the continuity equation we thus obtain

$$\frac{d\rho}{dt} = -\rho\nabla \cdot \boldsymbol{u} = 0 \qquad \Longrightarrow \qquad \nabla \cdot \boldsymbol{u} = 0$$

that is, the velocity is solenoidal field. Incompressibility is a reasonable good assumption for liquids and it may also hold in the case of gases, if perturbations are considerably less than the speed of sound. If an object moves slowly, in fact, the air in front of it can get rid of the compression at a speed faster than the speed at which the object is trying to build up compression (Choudhuri). Air can therefore be regarded as incompressible as long as all the motions inside have velocities small compared to the sound speed.

The vorticity equation has the same form as in Eq. (27), which involves only two variables ($\boldsymbol{u}$ and $\boldsymbol{\omega}$) which are not independent. If the velocity $\boldsymbol{u}$ is given, then we can easily find the vorticity as $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. On the contrary, if $\boldsymbol{\omega}$ is known, we can use a well-known results from mathematical physics stating that that a vector field can be solved for if its divergence and curl are given[1]. This allows one to find $\boldsymbol{u}$ from $\boldsymbol{\omega}$.

[1] Helmholtz's theorem, also known as the fundamental theorem of vector calculus, states that any sufficiently smooth, rapidly decaying vector field in three dimensions can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal (divergence-free) vector field.