Linear and Nonlinear Waves

A. Mignone
Physics Department
Turin University
I. THE SCALAR ADVECTION EQUATION
The Advection Equation: Theory

• First order partial differential equation (PDE) in \((x,t)\):

\[
\frac{\partial U(x,t)}{\partial t} + a \frac{\partial U(x,t)}{\partial x} = 0
\]

• Hyperbolic PDE: information propagates across domain at \textit{finite speed} → method of characteristics

• Characteristic curves satisfy:

\[
\frac{dx}{dt} = a
\]

• Along each characteristics:

\[
\frac{dU}{dt} = \frac{\partial U}{\partial t} + a \frac{dx}{dt} \frac{\partial U}{\partial x} = 0
\]

→ The solution is constant along characteristic curves.
The Advection Equation: Theory

- for constant $a$: the characteristics are straight parallel lines and the solution to the PDE is a uniform shift of the initial profile:

$$U(x, t) = U(x - at, 0)$$

- The solution shifts to the right (for $a > 0$) or to the left ($a < 0$):
System of Equations: Theory

• We turn our attention to the system of equations (PDE)

\[ \frac{\partial \mathbf{q}}{\partial t} + A \cdot \frac{\partial \mathbf{q}}{\partial x} = 0 \]

where \( \mathbf{q} = \{q_1, q_2, \ldots, q_m\} \) is the vector of unknowns. \( A \) is a \( m \times m \) constant matrix.

• For example, for \( m=3 \), one has

\[ \frac{\partial q_1}{\partial t} + A_{11} \frac{\partial q_1}{\partial x} + A_{12} \frac{\partial q_2}{\partial x} + A_{13} \frac{\partial q_3}{\partial x} = 0 \]
\[ \frac{\partial q_2}{\partial t} + A_{21} \frac{\partial q_1}{\partial x} + A_{22} \frac{\partial q_2}{\partial x} + A_{23} \frac{\partial q_3}{\partial x} = 0 \]
\[ \frac{\partial q_3}{\partial t} + A_{31} \frac{\partial q_1}{\partial x} + A_{32} \frac{\partial q_2}{\partial x} + A_{33} \frac{\partial q_3}{\partial x} = 0 \]
The system is hyperbolic if $A$ has real eigenvalues, $\lambda^1 \leq ... \leq \lambda^m$ and a complete set of linearly independent right and left eigenvectors $r^k$ and $l^k$ ($r^j \cdot l^k = \delta_{jk}$) such that

\[
\begin{cases}
A \cdot r^k = \lambda^k r^k \\
l^k \cdot A = l^k \lambda^k 
\end{cases}
\text{for } k = 1, ..., m
\]

For convenience we define the matrices $\Lambda = \text{diag}(\lambda^k)$, and

\[
R = (r^1 | r^2 | ... | r^m), \quad L = R^{-1} = \begin{pmatrix}
(l^1)^T \\
(l^2)^T \\
... \\
(l^m)^T
\end{pmatrix}
\]

So that $A \cdot R = R \cdot \Lambda, L \cdot A = \Lambda \cdot L, L \cdot R = R \cdot L = I, L \cdot A \cdot R = \Lambda.$
The linear system can be reduced to a set of decoupled linear advection equations.

Multiply the original system of PDE’s by $L$ on the left:

$$L \cdot \left( \frac{\partial q}{\partial t} + A \cdot \frac{\partial q}{\partial x} \right) = L \cdot \frac{\partial q}{\partial t} + L \cdot A \cdot R \cdot L \cdot \frac{\partial q}{\partial x} = 0$$

Define the **characteristic variables** $w = L \cdot q$ so that

$$\frac{\partial w}{\partial t} + \Lambda \cdot \frac{\partial w}{\partial x} = 0$$

Since $\Lambda$ is diagonal, these equations are not coupled anymore.
System of Equations: Theory

• In this form, the system decouples into $m$ independent advection equations for the characteristic variables:

$$
\frac{\partial w}{\partial t} + \Lambda \cdot \frac{\partial w}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial w^k}{\partial t} + \lambda^k \cdot \frac{\partial w^k}{\partial x} = 0
$$

where $w^k = l^k \cdot q$ (k=1,2,...,m) is a characteristic variable.

• When $m=3$ one has, for instance:

$$
\begin{align*}
\frac{\partial w^1}{\partial t} + \lambda^1 \frac{\partial w^1}{\partial x} &= 0 \\
\frac{\partial w^2}{\partial t} + \lambda^2 \frac{\partial w^2}{\partial x} &= 0 \\
\frac{\partial w^3}{\partial t} + \lambda^3 \frac{\partial w^3}{\partial x} &= 0
\end{align*}
$$
System of Equations: Theory

- The $m$ advection equations can be solved independently by applying the standard solution techniques developed for the scalar equation.

- In particular, one can write the *exact analytical solution* for the $k$-th characteristic field as

$$\omega^k(x, t) = \omega^k(x - \lambda^k t, 0)$$

i.e., the initial profile of $\omega^k$ shifts with uniform velocity $\lambda^k$, and

$$\omega^k(x - \lambda^k t, 0) = l^k \cdot q(x - \lambda^k t, 0)$$

is the initial profile.

- The characteristics are thus constant along the curves $dx/dt = \lambda^k$
Once the solution in characteristic space is known, we can solve the original system via the inverse transformation:

\[ q(x, t) = R \cdot w(x, t) = \sum_{k=1}^{k=m} w^k(x, t)r^k = \sum_{k=1}^{k=m} w^k(x - \lambda^k t, 0)r^k \]

The characteristic variables are thus the coefficients of the right eigenvector expansion of \( q \).

The solution to the linear system reduces to a linear combination of \( m \) linear waves traveling with velocities \( \lambda^k \).

Expressing everything in terms of the original variables \( q \),

\[ q(x, t) = \sum_{k=1}^{k=m} l^k \cdot q(x - \lambda^k t, 0)r^k \]
III. NONLINEAR SCALAR HYPERBOLIC PDE: BURGER’S EQUATION
We turn our attention to the scalar conservation law

\[ \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \]

Where \( f(u) \) is, in general, a nonlinear function of \( u \).

To gain some insights on the role played by nonlinear effects, we start by considering the inviscid Burger’s equation:

\[ \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) = 0 \]
Nonlinear Advection Equation

• We can write Burger’s equation also as
  \[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \]

• In this form, Burger’s equation resembles the linear advection equation, except that the velocity is no longer constant but it is equal to the solution itself.

• The characteristic curve for this equation is
  \[ \frac{dx}{dt} = u(x, t) \implies \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} \frac{dx}{dt} = 0 \]

• \( u \) is constant along the curve \( \frac{dx}{dt} = u(x, t) \) \( \implies \) characteristics are again straight lines: values of \( u \) associated with some fluid element do not change as that element moves.
Nonlinear Advection Equation

- From \( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \)

one can predict that, higher values of \( u \) will propagate faster than lower values: this leads to a wave steepening, since upstream values will advance faster than downstream values.
• Indeed, at $t=1$ the wave profile will look like:

• the wave steepens...
Nonlinear Advection Equation

- If we wait more, we should get something like this:

- A multi-value functions ⚠️! → Clearly **NOT** physical!
Burger Equation: Shock Waves

- The correct physical solution is to place a discontinuity there: a *shock wave*.

- Since the solution is no longer smooth, the differential form is not valid anymore and we need to consider the *integral form*. 
Burger Equation: Shock Waves

- This is how the solution should look like:

- Such solutions to the PDE are called *weak solutions*. 
Let’s try to understand what happens by looking at the characteristics.

Consider two states initially separated by a jump at an interface:

Here, the characteristic velocities on the left are greater than those on the right.
Burger Equation: Shock Waves

• The characteristic will intersect, creating a shock wave:

\[ \lambda(u_L) > S > \lambda(u_R) \]  

This is called the entropy condition.
Nonlinear Advection Equation

- The shock speed $S$ can be found using the Rankine-Hugoniot jump conditions, obtained from the integral form of the equation:

\[ f(u_R) - f(u_L) = S(u_R - u_L) \]

- For Burger’s equation $f(u) = u^2/2$, one finds the shock speed as

\[ S = \frac{u_L + u_R}{2} \]
Burger Equation: Rarefaction Waves

• Let’s consider the opposite situation:

Here, the characteristic velocities on the left are smaller than those on the right.
• Now the characteristics will diverge:

• Putting a shock wave between the two states would be incorrect, since it would violate the entropy condition. Instead, the proper solution is a \textit{rarefaction wave}.
Burger Equation: Rarefaction Waves

• A rarefaction wave is a nonlinear wave that smoothly connects the left and the right state. It is an expansion wave.

• The head of the rarefaction moves at the speed $\lambda(u_R)$, whereas the tail moves at the speed $\lambda(u_L)$.

• The solution can only be self-similar and takes on the range of values between $u_L$ and $u_R$.

• The general condition for a rarefaction wave is $\lambda(u_L) < \lambda(u_R)$.

• Both rarefactions and shocks are present in the solutions to the Euler equation. Both waves are nonlinear.
Burger Equation: Riemann Solver

• These results can be used to write the general solution to the Riemann problem for Burger’s equation:

– If $u_L > u_R$ the solution is a discontinuity (*shock wave*). In this case

$$u(x, t) = \begin{cases} 
  u_L & \text{if } x - St < 0 \\
  u_R & \text{if } x - St > 0 
\end{cases}, \quad S = \frac{u_L + u_R}{2}$$

– If $u_L < u_R$ the solution is a *rarefaction wave*. In this case

$$u(x, t) = \begin{cases} 
  u_L & \text{if } x/t \leq u_L \\
  x/t & \text{if } u_L < x/t < u_R \\
  u_R & \text{if } x/t \geq u_R 
\end{cases}$$
Nonlinear Advection Equation

- Solutions look like for a rarefaction and a shock, respectively.
IV. NONLINEAR SYSTEMS OF CONSERVATION LAW
Nonlinear Systems

• Much of what is known about the numerical solution of hyperbolic systems of nonlinear equations comes from the results obtained in the linear case or simple nonlinear scalar equations.

• The key idea is to exploit the conservative form and assume the system can be locally “frozen” at each grid interface.

• However, this still requires the solution of the Riemann problem, which becomes increasingly difficult for complicated set of hyperbolic P.D.E.
Euler Equations

- System of conservation laws describing conservation of mass, momentum and energy:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \quad \text{(mass)} \\
\frac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v} \mathbf{v} + \mathbf{I} p] &= 0 \quad \text{(momentum)} \\
\frac{\partial E}{\partial t} + \nabla \cdot [(E + p) \mathbf{v}] &= 0 \quad \text{(energy)}
\end{align*}
\]

- Total energy density \( E \) is the sum of thermal + Kinetic terms:

\[
E = \rho \epsilon + \rho \frac{\mathbf{v}^2}{2}
\]

- Closure requires an Equation of State (EoS).

For an ideal gas one has

\[
\rho \epsilon = \frac{p}{\Gamma - 1}
\]
Euler Equations: Characteristic Structure

- The equations of gasdynamics can also be written in “quasi-linear” or primitive form. In 1D:

\[
\frac{\partial \mathbf{V}}{\partial t} + A \cdot \frac{\partial \mathbf{V}}{\partial x} = 0, \quad A = \begin{pmatrix}
  v_x & \rho & 0 \\
  0 & v_x & 1/\rho \\
  0 & \rho c_s^2 & v_x
\end{pmatrix}
\]

where \( \mathbf{V} = [\rho, v_x, p] \) is a vector of primitive variable, \( c_s = (\gamma p/\rho)^{1/2} \) is the adiabatic speed of sound.

- It is called “quasi-linear” since, differently from the linear case where we had \( A = \text{const} \), here \( A = A(V) \).
Euler Equations: Characteristic Structure

- The quasi-linear form can be used to find the eigenvector decomposition of the matrix $A$:

  $$
  \mathbf{r}^1 = \begin{pmatrix} 1 \\ -c_s/\rho \\ c_s^2 \end{pmatrix}, \quad \mathbf{r}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}^3 = \begin{pmatrix} 1 \\ c_s/\rho \\ c_s^2 \end{pmatrix}
  $$

- Associated to the eigenvalues:

  $$
  \lambda^1 = v_x - c_s, \quad \lambda^2 = v_x, \quad \lambda^3 = v_x + c_s
  $$

- These are the characteristic speeds of the system, i.e., the speeds at which information propagates. They tell us a lot about the structure of the solution.
By looking at the expressions for the right eigenvectors,

\[
\begin{align*}
\mathbf{r}^1 &= \begin{pmatrix} 1 \\ -c_s/\rho \\ c_s^2 \end{pmatrix}, & \mathbf{r}^2 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, & \mathbf{r}^3 &= \begin{pmatrix} 1 \\ c_s/\rho \\ c_s^2 \end{pmatrix}
\end{align*}
\]

we see that across waves 1 and 3, all variables jump. These are nonlinear waves, either shocks or rarefactions waves.

Across wave 2, only density jumps. Velocity and pressure are constant. This defines the contact discontinuity.

The characteristic curve associated with this linear wave is \(dx/dt = u\), and it is a straight line. Since \(v_x\) is constant across this wave, the flow is neither converging or diverging.
Euler Equations: Riemann Problem

- The solution to the Riemann problem looks like

- The outer waves can be either shocks or rarefactions.
- The middle wave is always a contact discontinuity.
- In total one has 4 unknowns: $\rho_L^*, \rho_R^*, v_x^*, p^*$, since only density jumps across the contact discontinuity.
Euler Equations: Riemann Problem

• Depending on the initial discontinuity, a total of 4 patterns can emerge from the solution: