Linear and Nonlinear Waves

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I. THE SCALAR ADVECTION EQUATION

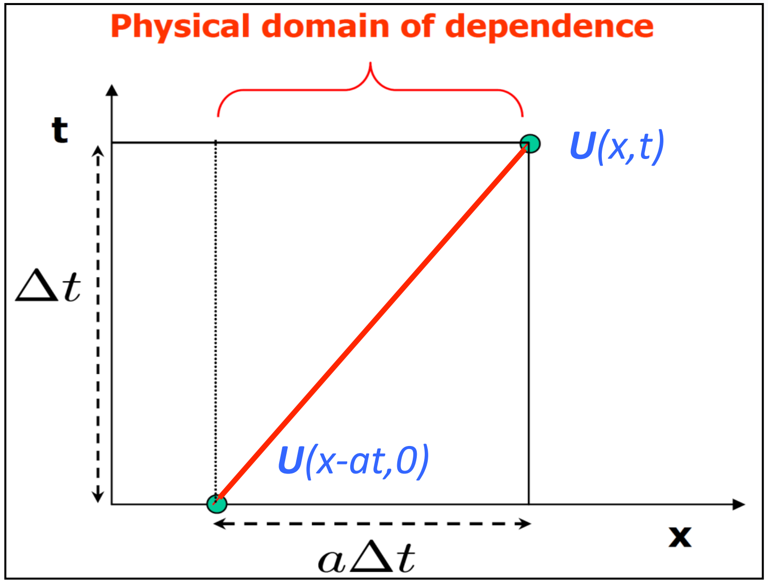
The Advection Equation: Theory

- First order partial differential equation (PDE) in (x,t):

$$\boxed{\frac{\partial U(x,t)}{\partial t} + a\frac{\partial U(x,t)}{\partial x} = 0}$$

- Hyperbolic PDE: information propagates across domain at *finite speed*
  → method of characteristics

- Characteristic curves satisfy: $\dfrac{dx}{dt} = a$



- Along each characteristics:

$$\frac{dU}{dt} = \frac{\partial U}{\partial t} + \frac{dx}{dt}\frac{\partial U}{\partial x} = 0$$

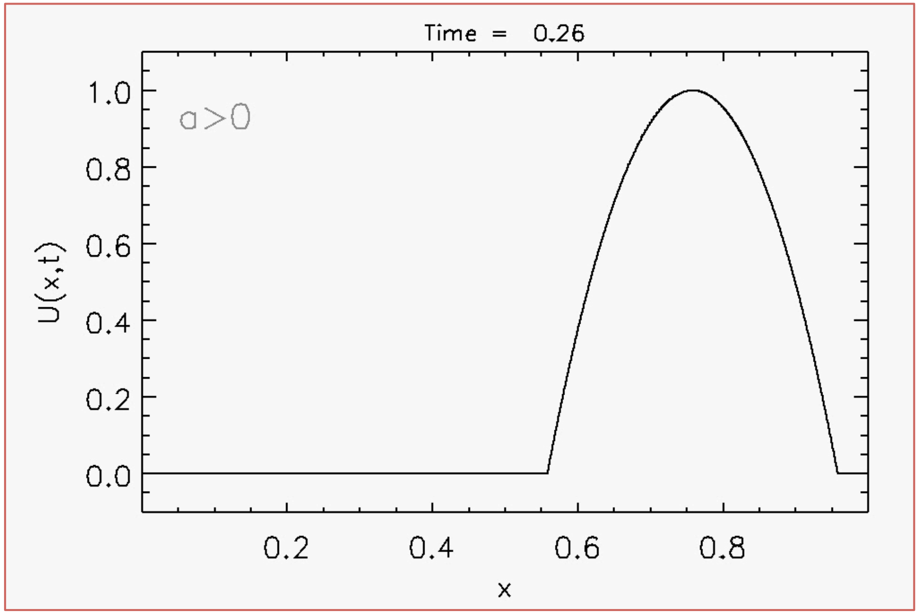
→ The solution is constant along characteristic curves.

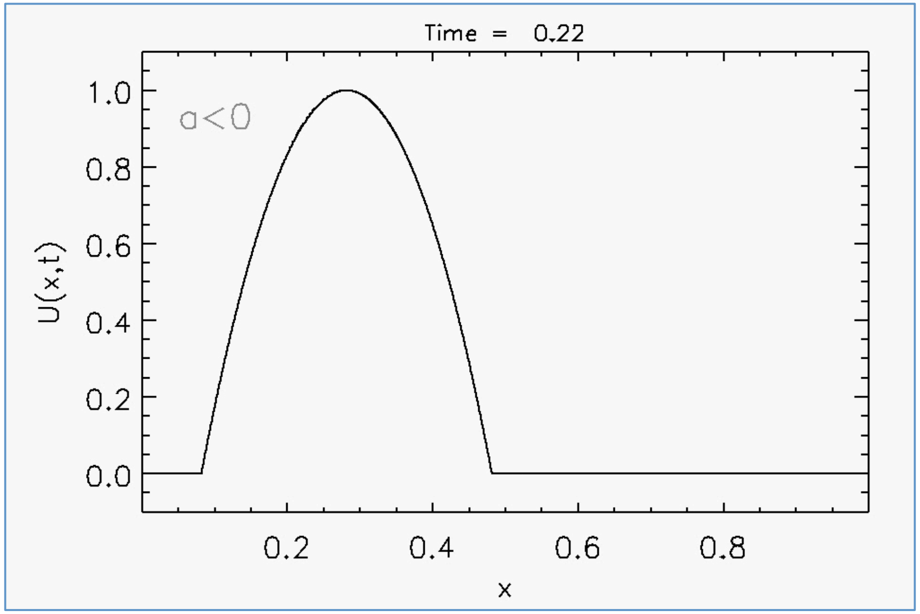
The Advection Equation: Theory

- for constant *a*: the characteristics are straight parallel lines and the solution to the PDE is a uniform shift of the initial profile:

$$U(x,t) = U(x-at,0)$$

- The solution shifts to the right (for $a > 0$) or to the left ($a < 0$):





II. LINEAR SYSTEMS OF HYPERBOLIC CONSERVATION LAWS

System of Equations: Theory

- We turn our attention to the system of equations (PDE)

$$\frac{\partial \mathbf{q}}{\partial t} + A \cdot \frac{\partial \mathbf{q}}{\partial x} = 0$$

  where $\mathbf{q} = \{q_1, q_2, \ldots q_m\}$ is the vector of unknowns. *A* is a $m \times m$ constant matrix.

- For example, for m=3, one has

$$\frac{\partial q_1}{\partial t} + A_{11}\frac{\partial q_1}{\partial x} + A_{12}\frac{\partial q_2}{\partial x} + A_{13}\frac{\partial q_3}{\partial x} = 0$$
$$\frac{\partial q_2}{\partial t} + A_{21}\frac{\partial q_1}{\partial x} + A_{22}\frac{\partial q_2}{\partial x} + A_{23}\frac{\partial q_3}{\partial x} = 0$$
$$\frac{\partial q_3}{\partial t} + A_{31}\frac{\partial q_1}{\partial x} + A_{32}\frac{\partial q_2}{\partial x} + A_{33}\frac{\partial q_3}{\partial x} = 0$$

System of Equations: Theory

- The system is hyperbolic if A has real eigenvalues, $\lambda^1 \leq \dots \leq \lambda^m$ and a complete set of linearly independent right and left eigenvectors r^k and l^k ($r^j \cdot l^k = \delta_{jk}$) such that

$$\begin{cases} A \cdot \boldsymbol{r}^k = \lambda^k \boldsymbol{r}^k \\ \boldsymbol{l}^k \cdot A = \boldsymbol{l}^k \lambda^k \end{cases} \qquad \text{for} \quad k = 1, \dots, m$$

- For convenience we define the matrices $\Lambda = diag(\lambda^k)$, and

$$R = (r^1 | r^2 | \dots | r^m)\,, \qquad L = R^{-1} = \begin{pmatrix} (l^1)^\intercal \\ \\ (l^2)^\intercal \\ \dots \\ (l^m)^\intercal \end{pmatrix}$$

- So that $A \cdot R = R \cdot \Lambda$, $L \cdot A = \Lambda \cdot L$, $L \cdot R = R \cdot L = I$, $L \cdot A \cdot R = \Lambda$.

System of Equations: Theory

- The linear system can be reduced to a set of decoupled linear advection equations.
- Multiply the original system of PDE’s by *L* on the left:

$$L \cdot \left(\frac{\partial \boldsymbol{q}}{\partial t} + A \cdot \frac{\partial \boldsymbol{q}}{\partial x} \right) = L \cdot \frac{\partial \boldsymbol{q}}{\partial t} + L \cdot A \cdot R \cdot L \cdot \frac{\partial \boldsymbol{q}}{\partial x} = 0$$

- Define the *characteristic variables* $w = L \cdot q$ so that

$$\boxed{\frac{\partial \boldsymbol{w}}{\partial t} + \Lambda \cdot \frac{\partial \boldsymbol{w}}{\partial x} = 0}$$

- Since Λ is diagonal, these equations are not coupled anymore.

System of Equations: Theory

- In this form, the system decouples into *m* independent advection equations for the characteristic variables:

$$\frac{\partial \boldsymbol{w}}{\partial t} + \Lambda \cdot \frac{\partial \boldsymbol{w}}{\partial x} = 0 \quad \Longrightarrow \quad \frac{\partial w^k}{\partial t} + \lambda^k \cdot \frac{\partial w^k}{\partial x} = 0$$

where $w^k = \mathbf{l}^k \cdot \mathbf{q}$ (k=1,2,…,m) is a characteristic variable.

- When m=3 one has, for instance:

$$\frac{\partial w^1}{\partial t} + \lambda^1 \frac{\partial w^1}{\partial x} = 0$$

$$\frac{\partial w^2}{\partial t} + \lambda^2 \frac{\partial w^2}{\partial x} = 0$$

$$\frac{\partial w^3}{\partial t} + \lambda^3 \frac{\partial w^3}{\partial x} = 0$$

System of Equations: Theory

- The m advection equations can be solved independently by applying the standard solution techniques developed for the scalar equation.

- In particular, one can write the *exact analytical solution* for the k-th characteristic field as

$$w^k(x,t) = w^k(x - \lambda^k t, 0)$$

i.e., the initial profile of w^k shifts with uniform velocity λ^k, and

$$w^k(x - \lambda^k t, 0) = \mathbf{l}^k \cdot \mathbf{q}(x - \lambda^k t, 0)$$

is the initial profile.

- The characteristics are thus constant along the curves $dx/dt = \lambda^k$

System of Equations: Exact Solution

- Once the solution in characteristic space is known, we can solve the original system via the inverse transformation

$$\mathbf{q}(x,t) = R \cdot \mathbf{w}(x,t) = \sum_{k=1}^{k=m} w^k(x,t)\mathbf{r}^k = \sum_{k=1}^{k=m} w^k(x-\lambda^k t, 0)\mathbf{r}^k$$

- The characteristic variables are thus the coefficients of the right eigenvector expansion of *q*.
- The solution to the linear system reduces to a linear combination of *m* linear waves traveling with velocities λ^k .
- Expressing everything in terms of the original variables *q*,

$$\boxed{\mathbf{q}(x,t) = \sum_{k=1}^{k=m} \mathbf{l}^k \cdot \mathbf{q}(x-\lambda^k t, 0)\mathbf{r}^k}$$

III. NONLINEAR SCALAR HYPERBOLIC PDE: BURGER'S EQUATION

Nonlinear Advection Equation

- We turn our attention to the scalar conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$$

- Where *f(u)* is, in general, a nonlinear function of *u*.

- To gain some insights on the role played by nonlinear effects, we start by considering the inviscid Burger's equation:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$$

Nonlinear Advection Equation

- We can write Burger's equation also as $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$

- In this form, Burger's equation resembles the linear advection equation, except that the velocity is no longer constant but it is equal to the solution itself.
- The characteristic curve for this equation is

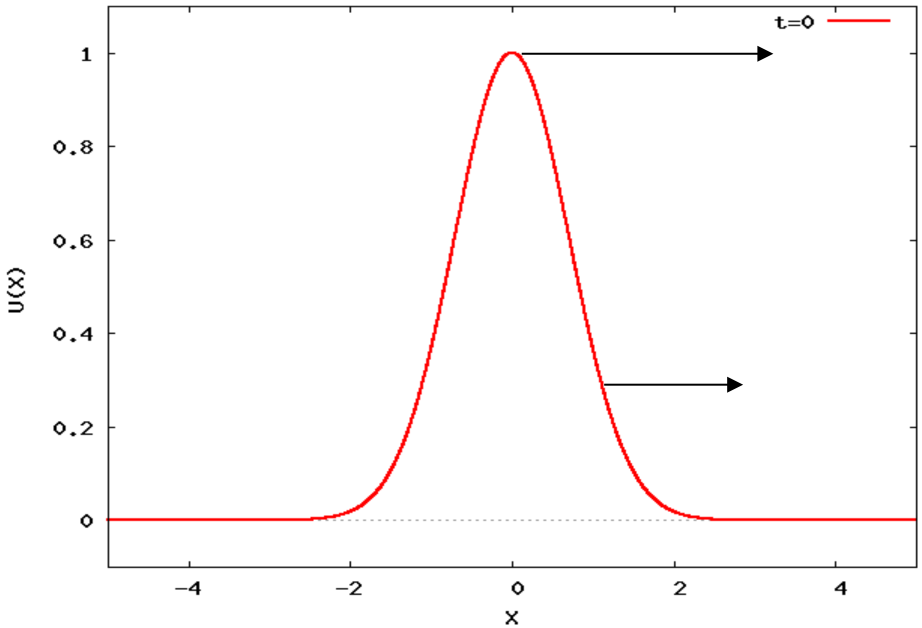
$$\frac{dx}{dt} = u(x,t) \quad \Longrightarrow \quad \frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = 0$$

- → *u* is constant along the curve *dx/dt=u(x,t)* → characteristics are again straight lines: values of *u* associated with some fluid element do not change as that element moves.

Nonlinear Advection Equation

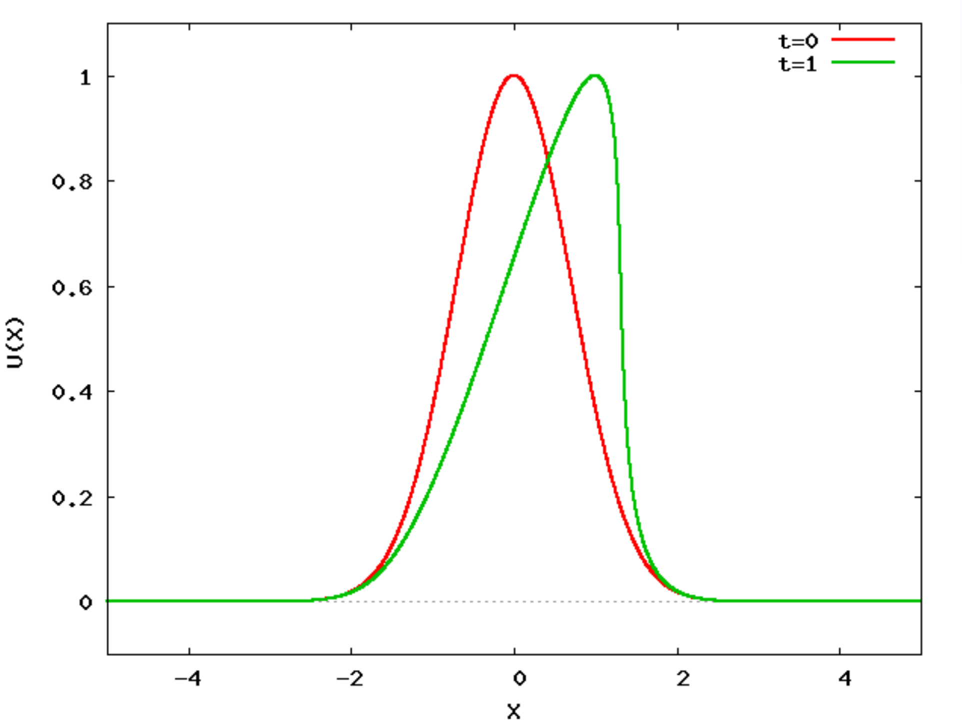
- From $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

one can predict that, higher values of *u* will propagate faster than lower values: this leads to a *wave steepening*, since upstream values will advances faster than downstream values.



Nonlinear Advection Equation

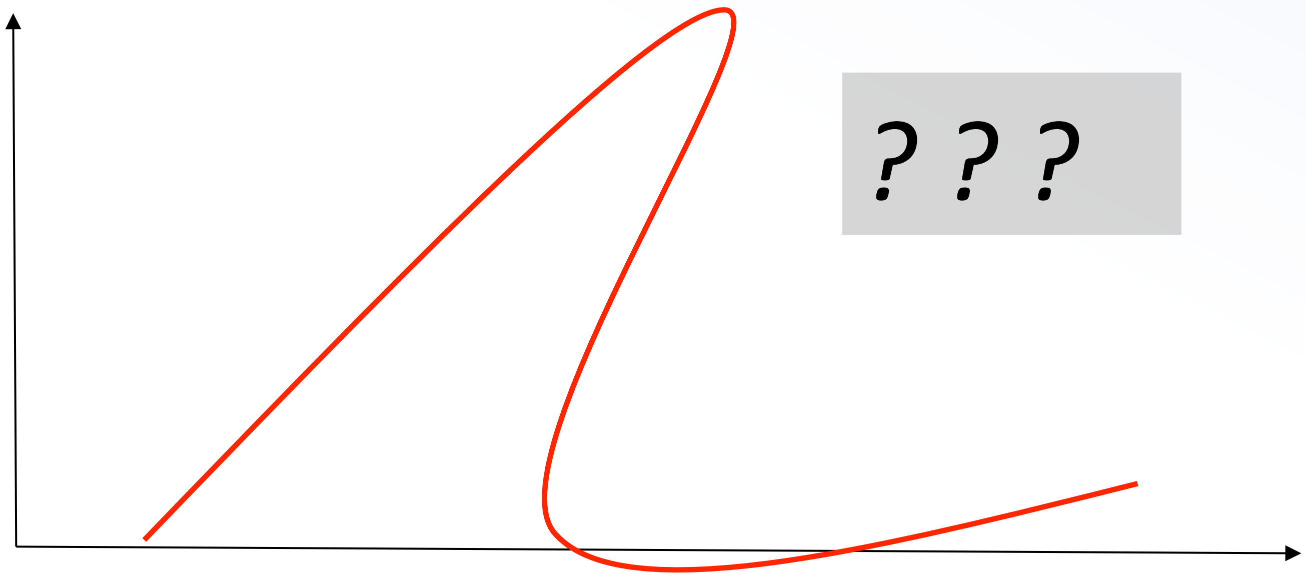
- Indeed, at t=1 the wave profile will look like:



- the wave steepens…

Nonlinear Advection Equation

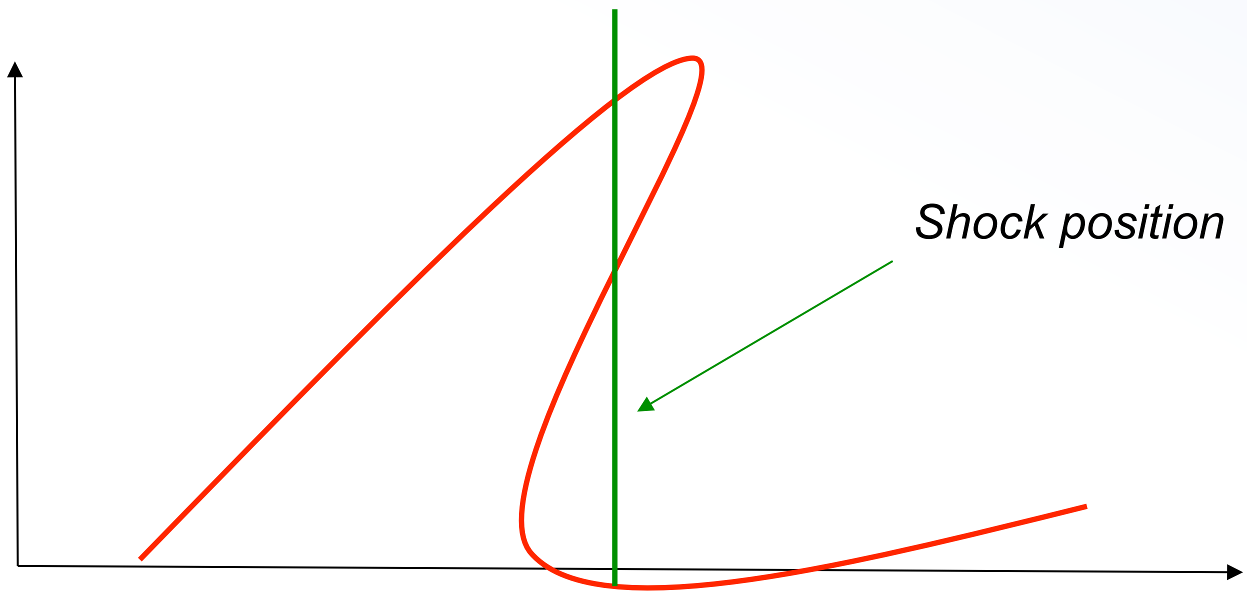
- If we wait more, we should get something like this:



- A multi-value functions ?! → Clearly *NOT* physical !

Burger Equation: Shock Waves

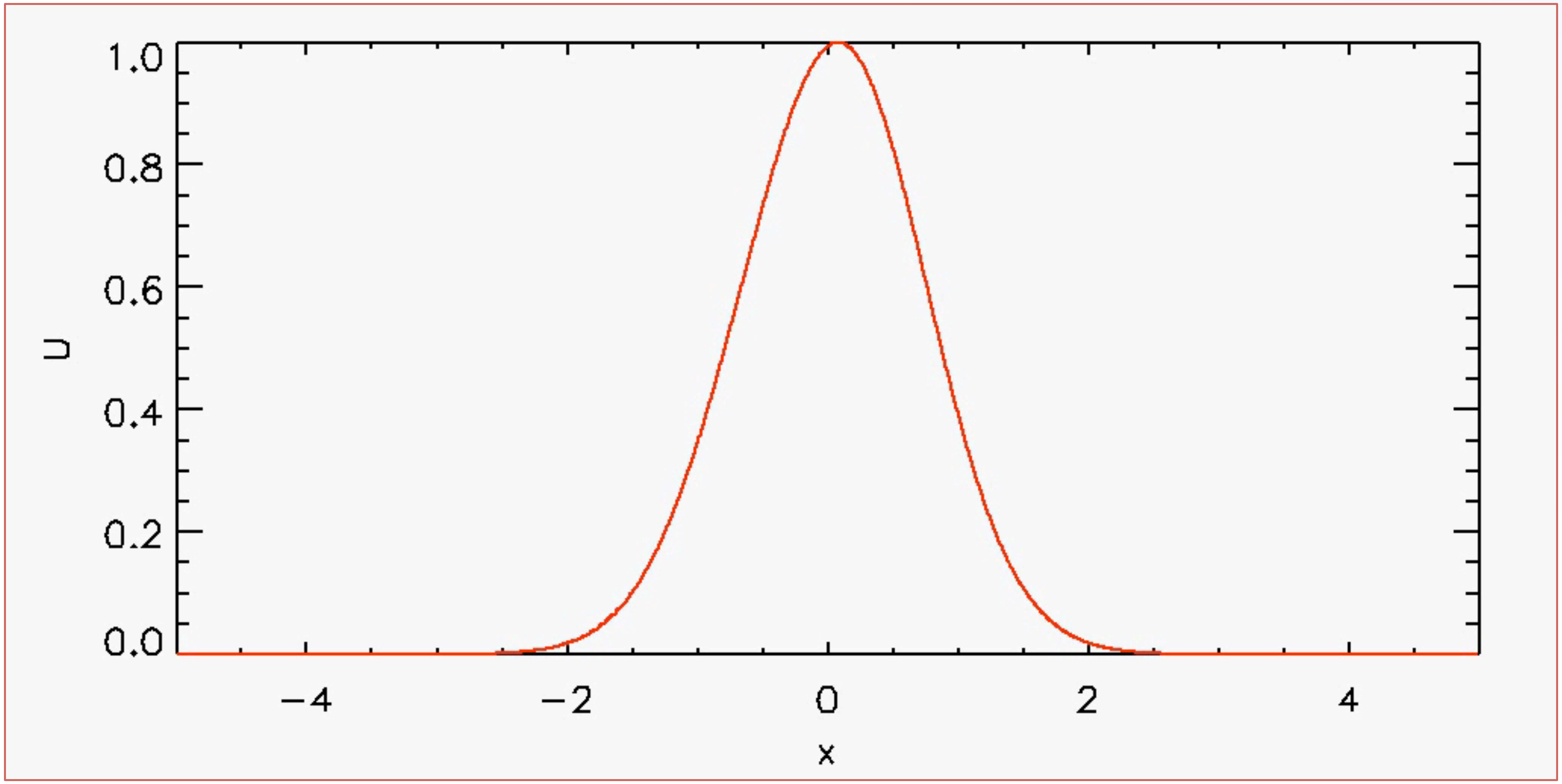
- The correct physical solution is to place a discontinuity there: a *shock wave*.



- Since the solution is no longer smooth, the differential form is not valid anymore and we need to consider the *integral form*.

Burger Equation: Shock Waves

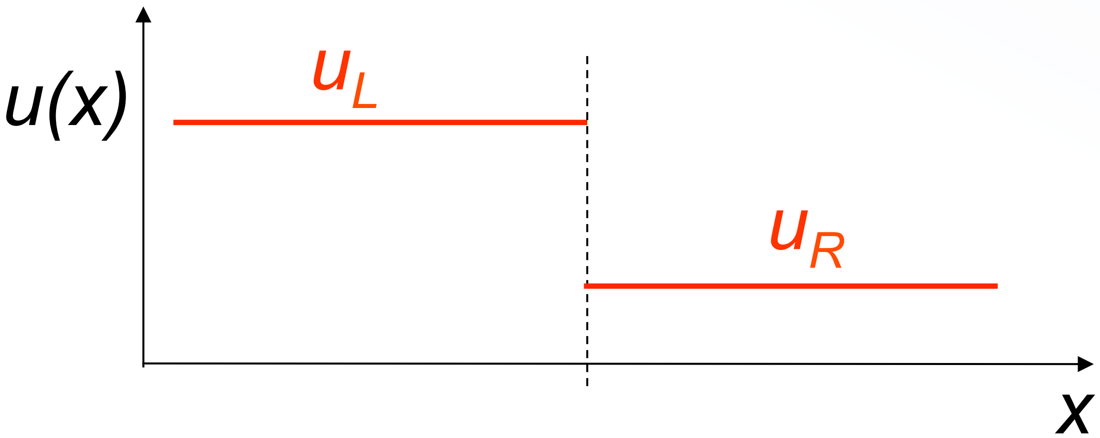
- This is how the solution should look like:



- Such solutions to the PDE are called *weak solutions*.

Burger Equation: Shock Waves

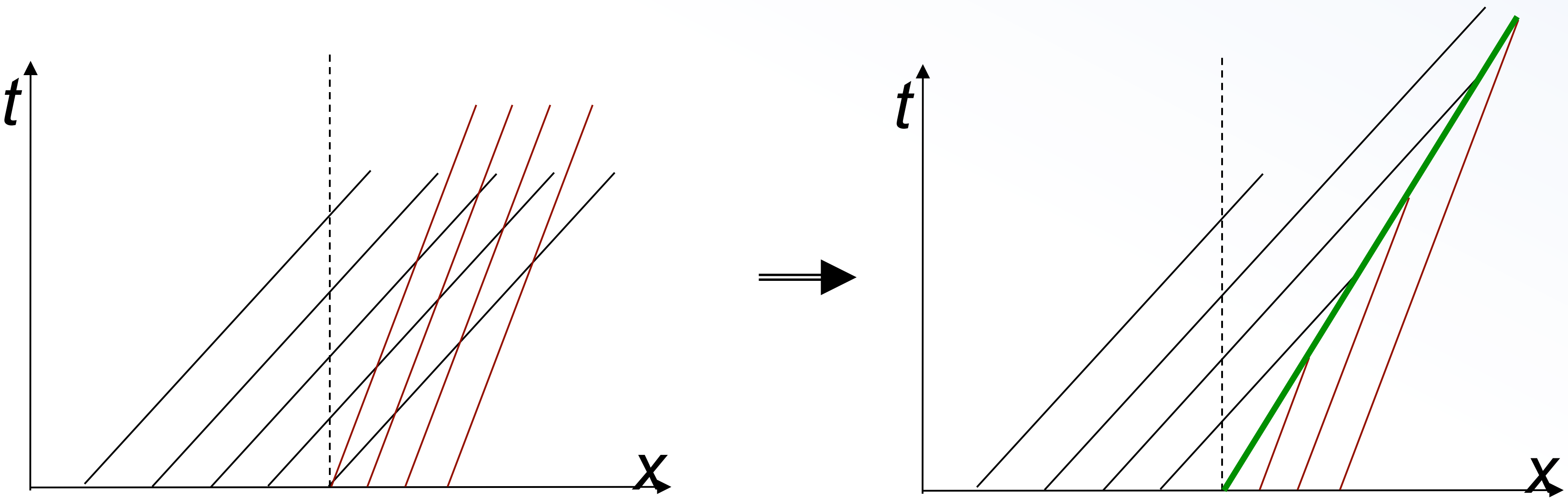
- Let's try to understand what happens by looking at the characteristics.
- Consider two states initially separated by a jump at an interface:



- Here, the characteristic velocities on the left are greater than those on the right.

Burger Equation: Shock Waves

- The characteristic will intersect, creating a *shock wave*:



- The shock speed is such that $\lambda(u_L) > S > \lambda(u_R)$. This is called the *entropy condition*.

Nonlinear Advection Equation

- The shock speed S can be found using the Rankine-Hugoniot jump conditions, obtained from the integral form of the equation:

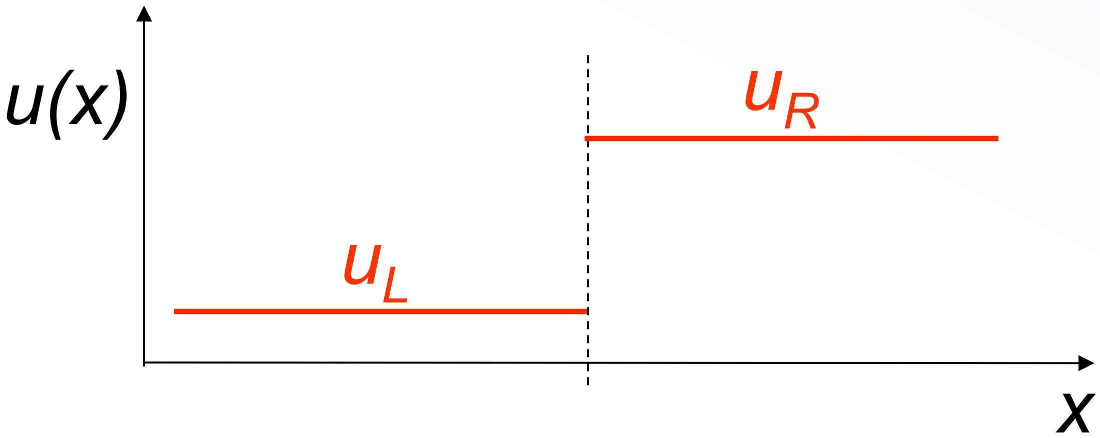
$$f(u_R) - f(u_L) = S(u_R - u_L)$$

- For Burger's equation $f(u) = u^2/2$, one finds the shock speed as

$$S = \frac{u_L + u_R}{2}$$

Burger Equation: Rarefaction Waves

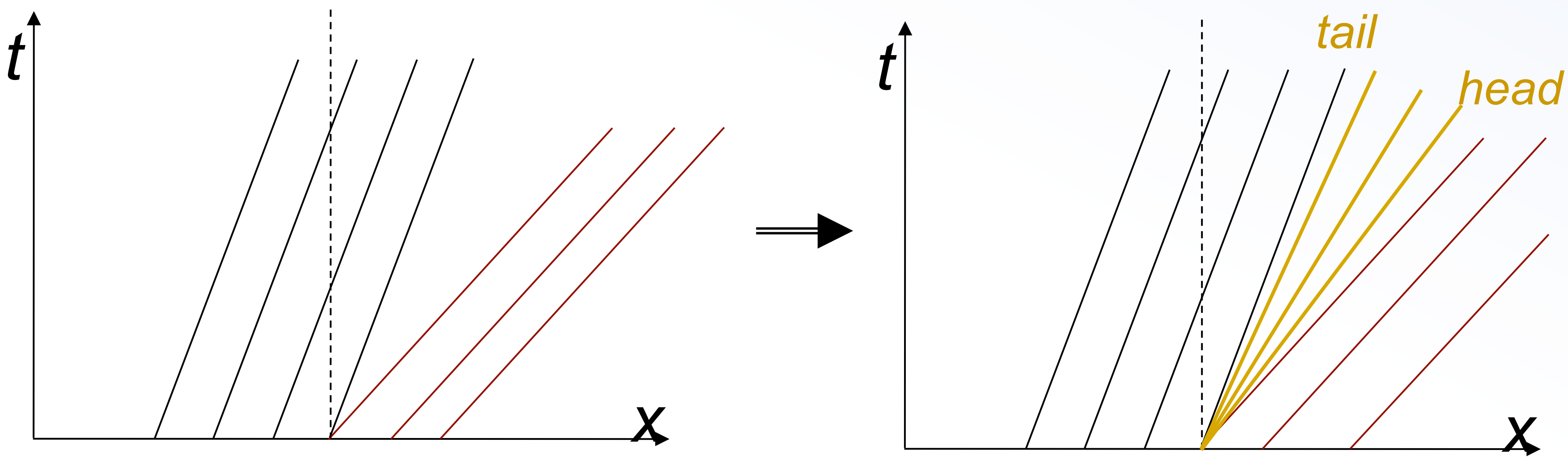
- Let's consider the opposite situation:



- Here, the characteristic velocities on the left are smaller than those on the right.

Burger Equation: Rarefaction Waves

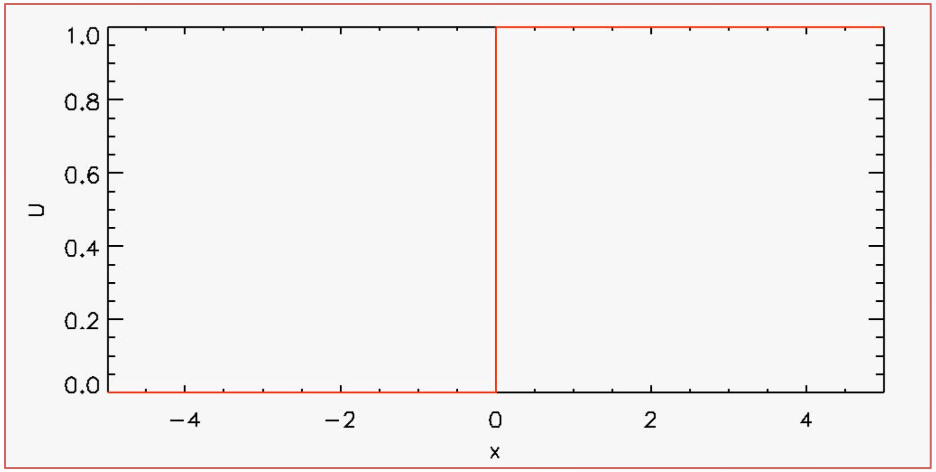
- Now the characteristics will diverge:



- Putting a shock wave between the two states would be incorrect, since it would violate the entropy condition. Instead, the proper solution is a *rarefaction wave*.

Burger Equation: Rarefaction Waves

- A rarefaction wave is a nonlinear wave that smoothly connects the left and the right state. It is an expansion wave.



- The solution can only be self-similar and takes on the range of values between u_L and u_R.

- The head of the rarefaction moves at the speed $\lambda(u_R)$, whereas the tail moves at the speed $\lambda(u_L)$.

- The general condition for a rarefaction wave is $\lambda(u_L)<\lambda(u_R)$

- Both rarefactions and shocks are present in the solutions to the Euler equation. Both waves are nonlinear.

Burger Equation: Riemann Solver

- These results can be used to write the general solution to the Riemann problem for Burger's equation:

  - If $u_L > u_R$ the solution is a discontinuity (*shock wave)*. In this case

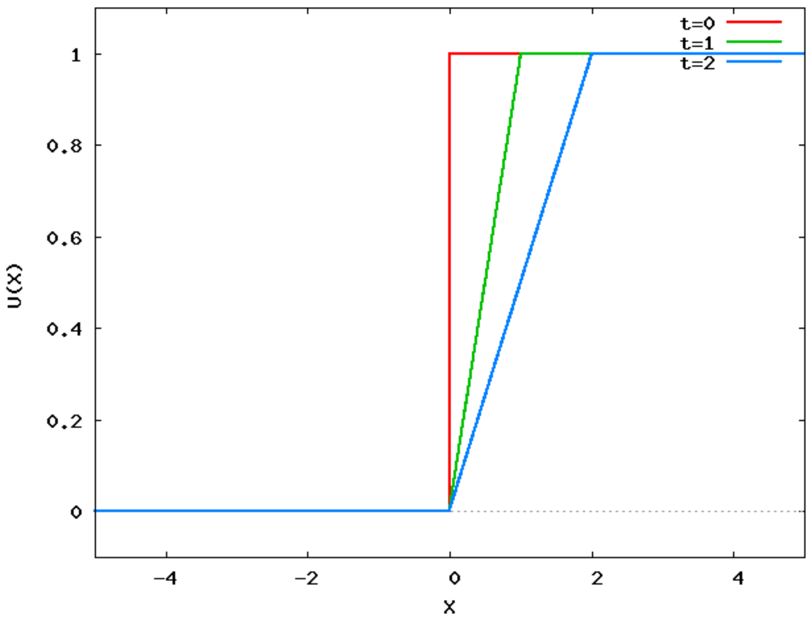
$$u(x,t) = \begin{cases} u_L & \text{if} \quad x - St < 0 \\ u_R & \text{if} \quad x - St > 0 \end{cases}, \qquad S = \frac{u_L + u_R}{2}$$

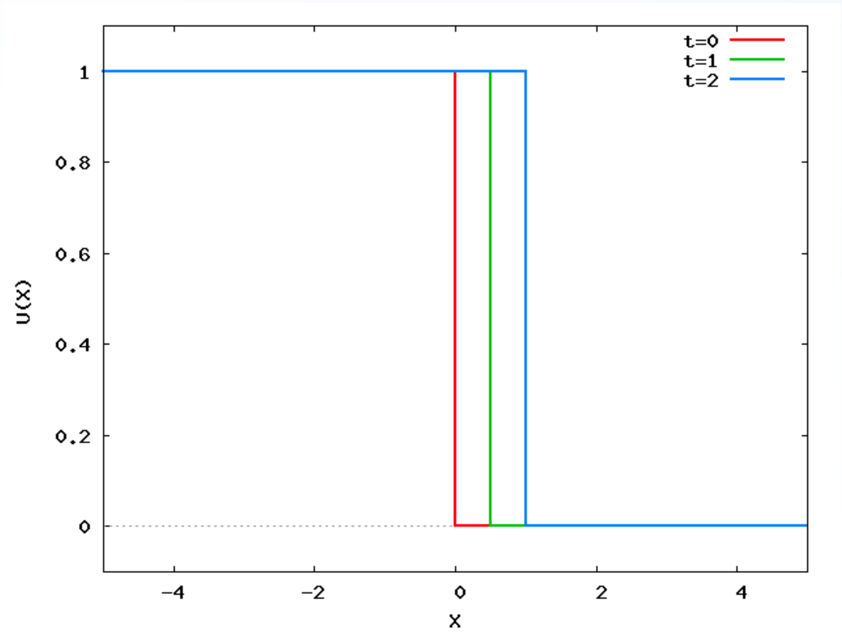
  - If $u_L < u_R$ the solution is a *rarefaction wave*. In this case

$$u(x,t) = \begin{cases} u_L & \text{if} \quad x/t \leq u_L \\ x/t & \text{if} \quad u_L < x/t < u_R \\ u_R & \text{if} \quad x/t > u_R \end{cases}$$

Nonlinear Advection Equation

- Solutions look like





- for a rarefaction and a shock, respectively.

IV. NONLINEAR SYSTEMS OF CONSERVATION LAW

Nonlinear Systems

- Much of what is known about the numerical solution of hyperbolic systems of nonlinear equations comes from the results obtained in the linear case or simple nonlinear scalar equations.

- The key idea is to exploit the conservative form and assume the system can be locally "frozen" at each grid interface.

- However, this still requires the solution of the Riemann problem, which becomes increasingly difficult for complicated set of hyperbolic P.D.E.

Euler Equations

- System of conservation laws describing conservation of mass, momentum and energy:

$$\boxed{\begin{array}{ll} \dfrac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 & \text{(mass)} \\ \dfrac{\partial (\rho \mathbf{v})}{\partial t} + \nabla \cdot [\rho \mathbf{v}\mathbf{v} + \mathbf{I} p] = 0 & \text{(momentum)} \\ \dfrac{\partial E}{\partial t} + \nabla \cdot [(E + p)\, \mathbf{v}] = 0 & \text{(energy)} \end{array}}$$

- Total energy density E is the sum of thermal + Kinetic terms:

$$E = \rho \epsilon + \rho \frac{\mathbf{v}^2}{2}$$

- Closure requires an Equation of State (EoS).

For an ideal gas one has $\rho \epsilon = \dfrac{p}{\Gamma - 1}$

Euler Equations: Characteristic Structure

- The equations of gasdynamics can also be written in "quasi-linear" or primitive form. In 1D:

$$\frac{\partial \mathbf{V}}{\partial t} + A \cdot \frac{\partial \mathbf{V}}{\partial x} = 0\,, \quad A = \begin{pmatrix} v_x & \rho & 0 \\ 0 & v_x & 1/\rho \\ 0 & \rho c_s^2 & v_x \end{pmatrix}$$

where $V = [\rho, v_x, p]$ is a vector of primitive variable, $c_s = (\gamma p/\rho)^{1/2}$ is the adiabatic speed of sound.

- It is called "quasi-linear" since, differently from the linear case where we had $A = const$, here $A = A(V)$.

Euler Equations: Characteristic Structure

- The quasi-linear form can be used to find the eigenvector decomposition of the matrix *A*:

$$\mathbf{r}^1 = \begin{pmatrix} 1 \\ -c_s/\rho \\ c_s^2 \end{pmatrix} , \quad \mathbf{r}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \mathbf{r}^3 = \begin{pmatrix} 1 \\ c_s/\rho \\ c_s^2 \end{pmatrix}$$

- Associated to the eigenvalues:

$$\lambda^1 = v_x - c_s \, , \quad \lambda^2 = v_x \, , \quad \lambda^3 = v_x + c_s$$

- These are the characteristic speeds of the system, i.e., the speeds at which information propagates. They tell us a lot about the structure of the solution.

Euler Equations: Riemann Problem

- By looking at the expressions for the right eigenvectors,

$$\mathbf{r}^1 = \begin{pmatrix} 1 \\ -c_s/\rho \\ c_s^2 \end{pmatrix}, \quad \mathbf{r}^2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{r}^3 = \begin{pmatrix} 1 \\ c_s/\rho \\ c_s^2 \end{pmatrix}$$

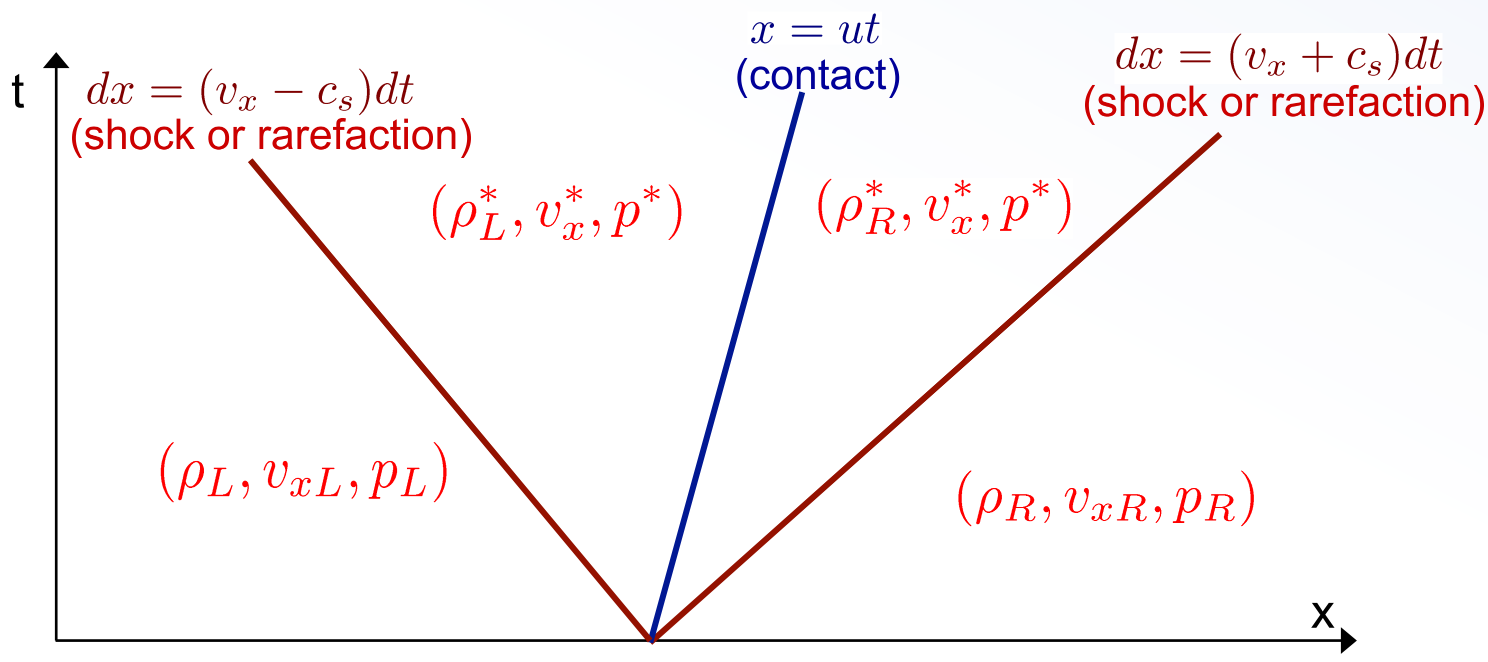
- we see that across waves 1 and 3, all variables jump. These are nonlinear waves, either shocks or rarefactions waves.

- Across wave 2, only density jumps. Velocity and pressure are constant. This defines the *contact discontinuity*.

- The characteristic curve associated with this linear wave is $dx/dt = u$, and it is a straight line. Since v_x is constant across this wave, the flow is neither converging or diverging.

Euler Equations: Riemann Problem

- The solution to the Riemann problem looks like



- The outer waves can be either shocks or rarefactions.
- The middle wave is always a contact discontinuity.
- In total one has 4 unknowns: $\rho_L^*, \rho_R^*, v_x^*, p^*$, since only density jumps across the contact discontinuity.

Euler Equations: Riemann Problem

- Depending on the initial discontinuity, a total of 4 patterns can emerge from the solution: