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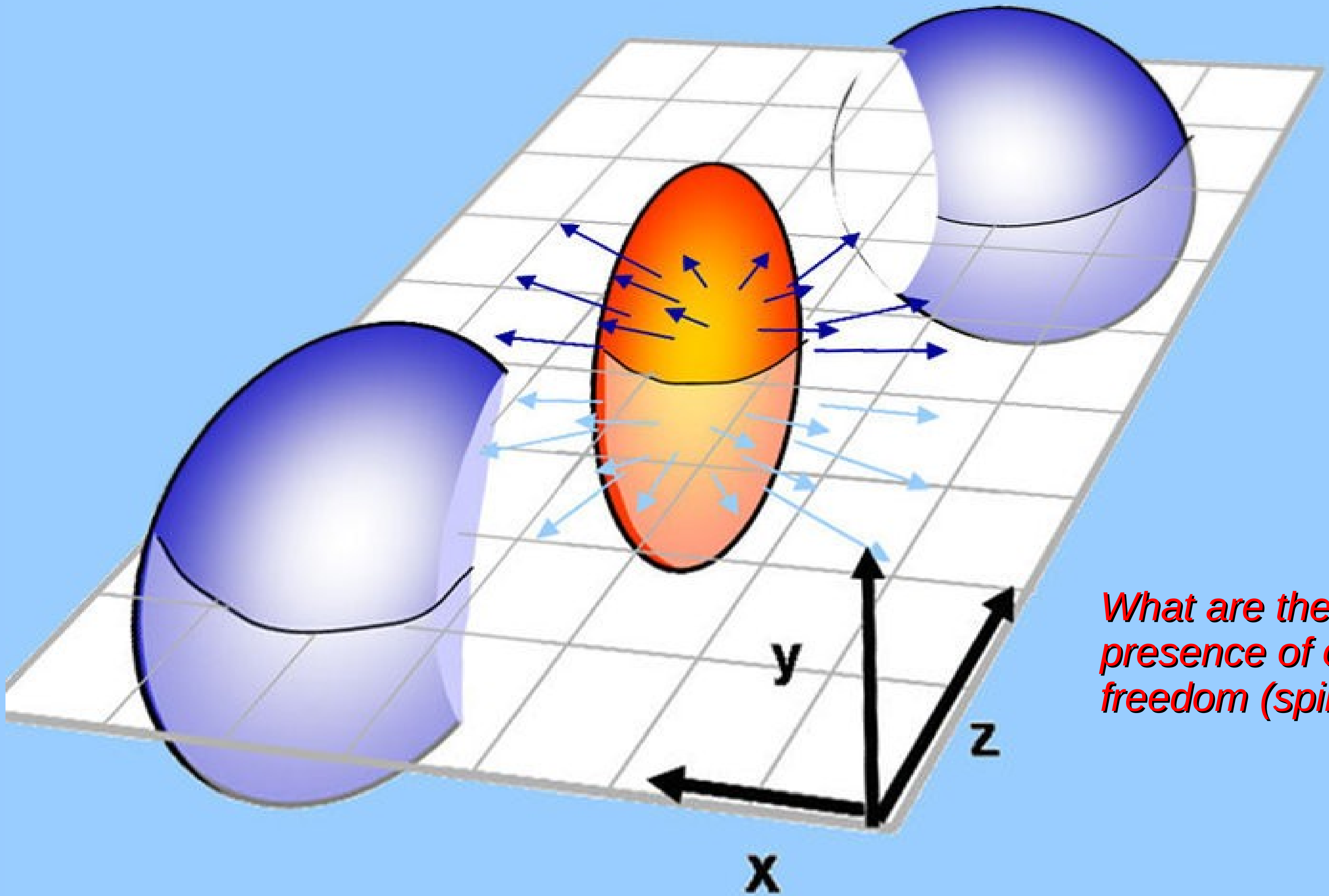
Relativistic fluids with spin

Outline

- Introduction
- Inequivalence of stress-energy and spin tensor
- Relativistic fluids at thermodynamical equilibrium
- The classical limit
- Ongoing investigations

Motivations

The success of the hydrodynamical description of Quark Gluon Plasma



What are the consequences of the presence of other degrees of freedom (spin)?

Fluids with spin: from classical to relativistic formulation

A non relativistic fluid with spin has a total linear momentum (constant for isolated system), and a momentum density fulfilling:

$$\boldsymbol{P} = \int_V \boldsymbol{p}(\boldsymbol{x}, t) d^3\boldsymbol{x}, \qquad \partial_t \boldsymbol{p}(\boldsymbol{x}, t) + \nabla T(\boldsymbol{x}, t) = 0$$

The angular momentum naturally gets a spin term:

$$\begin{cases} \boldsymbol{l}(\boldsymbol{x}, t) = \boldsymbol{x} \times \boldsymbol{p}(\boldsymbol{x}, t) \\ \boldsymbol{j}(\boldsymbol{x}, t) = \boldsymbol{l}(\boldsymbol{x}, t) + \boldsymbol{s}(\boldsymbol{x}, t) \end{cases} \qquad \boldsymbol{J} = \int_V \boldsymbol{j}(\boldsymbol{x}, t) d^3\boldsymbol{x}$$

For particles we have the following changes

$$\boldsymbol{p} \longrightarrow p^\mu$$
$$\boldsymbol{l} \longrightarrow l^{\mu\nu} = x^\mu p^\nu - p^\mu x^\nu$$
$$\boldsymbol{s} \longrightarrow s^{\mu\nu}$$

Likewise densities become

$$\boldsymbol{p}(\boldsymbol{x}, t) \longrightarrow T^{0\mu}(x)$$
$$\boldsymbol{l}(\boldsymbol{x}, t) \longrightarrow x^\mu T^{0\nu}(x) - T^{0\mu}(x) x^\nu$$
$$\boldsymbol{s}(\boldsymbol{x}, t) \longrightarrow \mathcal{S}^{0,\mu\nu}(x)$$

For covariance the momentum density follows the equation:

$$\partial_t T^{0\mu} + \partial_i T^{i\mu} = 0$$

What is a relativistic fluid with spin?

A fluid with spin is a fluid which is described by a spin tensor $\mathcal{S}^{\lambda,\mu\nu}$ *besides the stress-energy tensor* $T^{\mu\nu}$ *:*

$$\int_V T^{0\mu} d^3\boldsymbol{x} = P^\mu$$

Total four-momentum

$$\int_V \left(\mathcal{S}^{0,\mu\nu} + x^\mu T^{0\nu} - x^\nu T^{0\mu}\right) d^3\boldsymbol{x} = J^{\mu\nu}$$

Total angular momentum

Four-momentum and angular momentum conservation equations:

$$\begin{cases} \partial_\mu T^{\mu\nu} = 0 \\ \partial_\lambda \mathcal{S}^{\lambda,\mu\nu} = T^{\nu\mu} - T^{\mu\nu} \end{cases}$$

Fluidodynamic models

- J. Weyssenhoff, A. Raabe, Acta Phys. Pol. 9, 7 (1947).

$$\mathcal{S}^{\lambda,\mu\nu} = u^{\lambda}\sigma^{\mu\nu}$$

Frenkel condition $$t^{\mu} = \sigma^{\mu\nu}u_{\nu} = 0$$

- D. Bohm and J.P. Vigier, Phys. Rev. 109, 1882 (1958)

  Critique (well founded) of the Frenkel condition

- F. Halbwachs, Theorie relativiste des fluids `a spin, Gauthier-Villars, Paris, (1960)

  Variational lagrangian theory of ideal fluids with spin, still embodying Frenkel condition

From 1960 onwards: many papers with extension and possible applications of Halbwach's theory to general relativity, Einstein- Cartan theory and relevant cosmologies

Is it really the right formulation?

An ideal rotating Boltzmann gas at full thermodynamical equilibrium doesn't fulfill the Frenkel condition!

- F.B., L. Tinti, Ann. Phys. 325, 1566 (2010)

From quantum mechanics to hydrodynamics

Mean values:

$$\mathcal{O}_{\mathrm{cl.}} = \mathrm{tr}\left(\widehat{\rho}\widehat{O}\right)$$

Therefore:

$$T^{\mu\nu}(x) = \mathrm{tr}\left(\widehat{\rho}\,\widehat{T}^{\mu\nu}(x)\right) \quad \mathcal{S}^{\lambda,\mu\nu}(x) = \mathrm{tr}\left(\widehat{\rho}\,\widehat{\mathcal{S}}^{\lambda,\mu\nu}(x)\right)$$

But in quantum field theory (without gravity) stress-energy and spin tensors are not uniquely defined.

So, what operators do we use?

Do the microscopic operators give us the same macroscopic results?

Thermodynamical inequivalence of quantum stress-energy and spin tensors

F. Becattini, L. Tinti, arXiv:1101.5251v2

In quantum field theory there is a class of transformations

$$\widehat{T'}^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{1}{2}\partial_\alpha\left(\widehat{\Phi}^{\alpha,\mu\nu} - \widehat{\Phi}^{\mu,\alpha\nu} - \widehat{\Phi}^{\nu,\alpha\mu}\right)$$

$$\widehat{\mathcal{S}'}^{\lambda,\mu\nu} = \widehat{\mathcal{S}}^{\lambda,\mu\nu} - \widehat{\Phi}^{\lambda,\mu\nu}$$

Giving the same generators of the Poincaré group, provided that the boundary integrals vanish:

$$\int_{\partial V} dS \left(\widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{\nu,0i}\right) n_i = 0$$

$$\int_{\partial V} dS \left[x^\mu \left(\widehat{\Phi}^{0,i\nu} - \widehat{\Phi}^{i,0\nu} - \widehat{\Phi}^{\nu,0i}\right) - x^\nu \left(\widehat{\Phi}^{0,i\mu} - \widehat{\Phi}^{i,0\mu} - \widehat{\Phi}^{\mu,0i}\right)\right] n_i = 0$$

Requiring that energy, momentum and total angular momentum densities macroscopically take on objective values:

or, in a covariant form

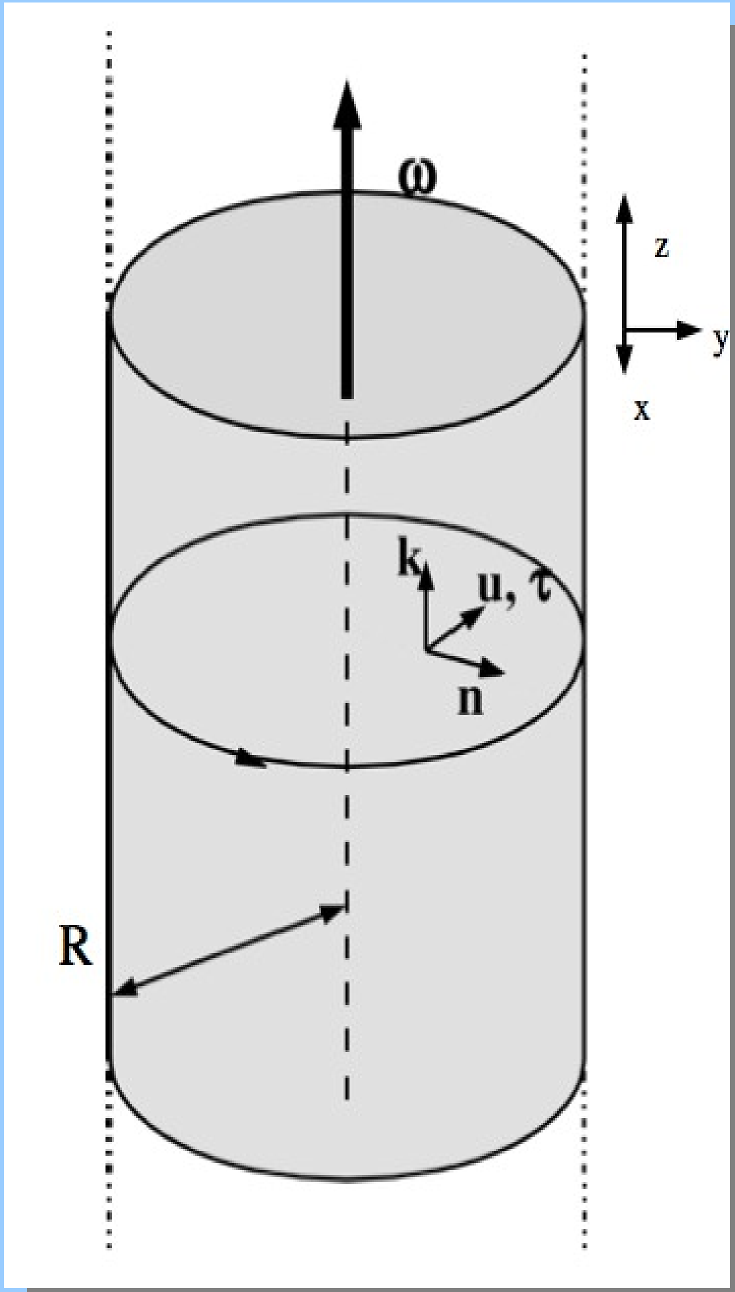
$$T'^{\mu\nu} = T^{\mu\nu}$$

$$\mathcal{J}'^{\lambda,\mu\nu} = \mathcal{J}^{\lambda,\mu\nu} + g^{\lambda\mu}K^\nu - g^{\lambda\nu}K^\mu$$

$$\Rightarrow \partial^\nu K^\mu = 0$$

When does this inequivalence matter?

The equivalence between different microscopic tensor depends on the state of the system, in particular on its symmetries.



e.g. **in the grand-canonical ensamble every microscopic tensor give the same average momentum and angular momentum density.**

What happens in a less symmetric case?

$$\widehat{\rho} = \frac{1}{Z_\omega} \mathrm{P}_V \exp(-\widehat{H}/T + \boldsymbol{\omega} \cdot \widehat{\boldsymbol{J}}/T + \mu \widehat{Q}/T)$$

$$Z_\omega = \mathrm{tr}[\mathrm{P}_V \exp(-\widehat{H}/T + \boldsymbol{\omega} \cdot \widehat{\boldsymbol{J}}/T + \mu \widehat{Q}/T)]$$

$$u = (\gamma, \gamma \mathbf{v}) \qquad \tau = (\gamma v, \gamma \hat{\mathbf{v}})$$

$$n = (0, \hat{\mathbf{r}}) \qquad k = (0, \hat{\mathbf{k}})$$

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$$

Example: Free rotating Dirac field in a cylindrical region

The Lagrangian density give us the canonical tensor operator:

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m\overline{\Psi}\Psi \Rightarrow \qquad \begin{aligned} \widehat{T}^{\mu\nu} &= \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi \\ \widehat{\mathcal{S}}^{\lambda,\mu\nu} &= \frac{1}{2}\overline{\Psi}\{\gamma^\lambda, \Sigma^{\mu\nu}\}\Psi = \frac{i}{8}\overline{\Psi}\{\gamma^\lambda, [\gamma^\mu, \gamma^\nu]\}\Psi \end{aligned}$$

Are the tensor given by Belinfante symmetrization equivalent like in the grand-canonical ensemble?

$$\widehat{T}'^{\mu\nu} = \frac{i}{4}\left[\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial}^\nu \Psi + \overline{\Psi}\gamma^\nu \overleftrightarrow{\partial}^\mu \Psi\right]$$

$$\widehat{\mathcal{S}}'^{\lambda,\mu\nu} = 0$$

i.e. does the spin tensor operator represent a suitable transformation ?

Due to the additional symmetry of $\widehat{\mathcal{S}}^{\lambda,\mu\nu}$, the average value is much simpler than in the general case:

$$\mathcal{S}^{\lambda,\mu\nu} = D(r)[(n^\mu\tau^\nu - n^\nu\tau^\mu)u^\lambda + (n^\lambda\tau^\mu - n^\mu\tau^\lambda)u^\nu - (n^\lambda\tau^\nu - n^\nu\tau^\lambda)u^\mu]$$

In particular: $$\mathcal{S}^{0,12} = \mathrm{tr}\left(\widehat{\rho}\,\widehat{\mathcal{S}}^{0,12}\right) = D(r)$$

Free rotating Dirac field in a cylindrical region

$$T^{0i}_{\text{Belinfante}} = T^{0i}_{\text{canonical}} - \frac{1}{2}\frac{\mathrm{d}D(r)}{\mathrm{d}r}\hat{v}^i$$

$$\mathcal{J}_{\text{Belinfante}} = \mathcal{J}_{\text{canonical}} - \left(\frac{1}{2}r\frac{\mathrm{d}D(r)}{\mathrm{d}r} + D(r)\right)\hat{\mathbf{k}}$$

From an explicit calculation we found:

$$\mathrm{tr}[\widehat{\rho} : \Psi^\dagger(0,\mathbf{x})\Sigma_z\Psi(0,\mathbf{x}):] = D(r)$$

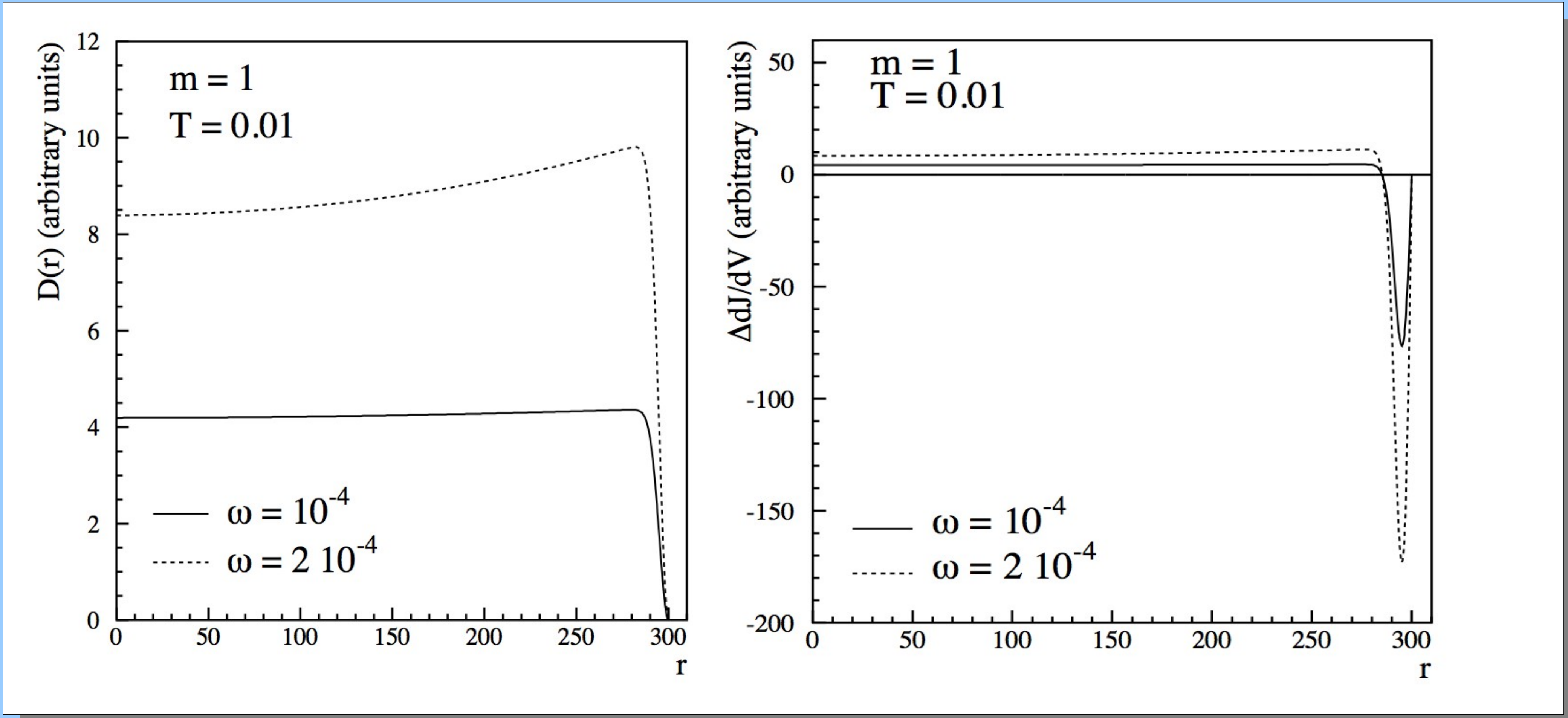
$$= \sum_M \sum_{\xi=\pm1} \sum_{l=1}^{\infty} \int_{-\infty}^{\infty} \mathrm{d}p_z \left[\frac{1}{\mathrm{e}^{(\varepsilon - M\omega + \mu)/T} + 1} + \frac{1}{\mathrm{e}^{(\varepsilon - M\omega - \mu)/T} + 1}\right] \frac{p_{Tl}^2\left[J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}r) - b_\xi^{(+)^2} J^2_{\left|M+\frac{1}{2}\right|}(p_{Tl}r)\right]}{4\pi R J^2_{\left|M-\frac{1}{2}\right|}(p_{Tl}R)(2Rm_{Tl}^2 + 2\xi M m_{Tl} + m)}$$

This is a difficult expression to handle, but it's more simple to study the behavior at $r = 0$

In fact $D(0) = 0$ when there is no angular velocity, but it become positive in a positive neighborhood of ω

The non relativistic limit

In the non-relativistic limit we can perform a numerical computation of the D(r) function:



Conclusion:

Canonical and Belinfante tensor have been proven to be inequivalent in general and in particular to give different momentum and angular momentum densities for a free Dirac field with non vanishing angular momentum.

Can we measure it ?

We have to find a proper way to measure the polarization of a rotating system along the axis.

- Barnett effect?
- Einstein-de Hass effect?
- Polarization of a rotating ultracold gas?

Work in progress: (NR) quantum kinetic approach

Goal: find the shape of stress-energy and spin tensors in a non-equilibrium quantum kinetic model to classify spin transport coefficients

Start from the simple non-relativistic case with fixed number of particles.

from:

$$i\hbar\partial_t\widehat{\rho} = [\widehat{H},\widehat{\rho}] \qquad \widehat{F}_1 \equiv N\mathrm{tr}_{N-1}(\widehat{\rho}) \;\; \widehat{F}_{12} \equiv (N-1)\mathrm{tr}_{N-2}(\widehat{\rho})$$

we get

$$i\hbar\partial_t\widehat{F}_1 = \left[\widehat{H}_1,\widehat{F}_1\right] + \left[\widehat{V}_{12},\widehat{F}_{12}\right]$$

At first order in the BBGKY hierarchy every one particle quantity can be expressed in relation with the matrix elements:

$$F_1(\boldsymbol{r}',\boldsymbol{r}'';t)^{\alpha}_{\beta} = \langle\boldsymbol{r}',\alpha|\,\widehat{F}_1(t)\,|\boldsymbol{r}'',\beta\rangle$$

Evolution of a fluid system out of equilibrium

With a change of variable and a Fourier transform we pass to the Wigner basis:

$$F_1(\boldsymbol{r}', \boldsymbol{r}''; t)^{\alpha}_{\beta} = \langle \boldsymbol{r}', \alpha | \, \widehat{F}_1(t) \, | \boldsymbol{r}'', \beta \rangle$$

$$\begin{cases} \boldsymbol{x} = \frac{1}{2}(\boldsymbol{r}' + \boldsymbol{r}'') \\ \boldsymbol{r} = \boldsymbol{r}' - \boldsymbol{r}'' \end{cases} , \begin{cases} \boldsymbol{r}' = \boldsymbol{x} + \frac{1}{2}\boldsymbol{r} \\ \boldsymbol{r}'' = \boldsymbol{x} - \frac{1}{2}\boldsymbol{r} \end{cases} \Rightarrow \begin{cases} \nabla_{\boldsymbol{r}'} = \frac{1}{2}\nabla_{\boldsymbol{x}} + \nabla_{\boldsymbol{r}} \\ \nabla_{\boldsymbol{r}''} = \frac{1}{2}\nabla_{\boldsymbol{x}} + \nabla_{\boldsymbol{r}} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} \nabla^2_{\boldsymbol{r}'} = \frac{1}{4}\nabla^2_{\boldsymbol{x}} + \nabla^2_{\boldsymbol{r}} + \nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{r}} \\ \nabla_{\boldsymbol{r}''} = \frac{1}{4}\nabla^2_{\boldsymbol{x}} + \nabla^2_{\boldsymbol{r}} - \nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{r}} \end{cases} \Rightarrow \nabla^2_{\boldsymbol{r}'} - \nabla^2_{\boldsymbol{r}''} = 2\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{r}}.$$

Wigner function:

$$f(\mathbf{x}, \mathbf{p}; t)^{\alpha}_{\beta} = \frac{1}{(2\pi\hbar)^3} \int d^3\boldsymbol{r} \; \mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}/\hbar} \tilde{F}_1(\boldsymbol{x}, \boldsymbol{r}; t)^{\alpha}_{\beta}$$

So average values of observables can be written in this form:

$$O_1(t) = \mathrm{tr}\left(\int d^3\boldsymbol{x} d^3\boldsymbol{p} \; \widehat{O}_1(\boldsymbol{x}, \boldsymbol{p}) f(\mathbf{x}, \mathbf{p}; t) \right)$$

where

$$\widehat{O}_1(\boldsymbol{x}, \boldsymbol{p})^{\beta}_{\alpha} = \int d^3\boldsymbol{r} \; \widehat{\widetilde{O}}_1(\boldsymbol{x}, \boldsymbol{r})^{\beta}_{\alpha} \mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar}$$

What we have found so far

The evolution equation of momentum density an spin density:

$$
\begin{aligned}
\frac{d}{dt}\boldsymbol{p}(\boldsymbol{x},t) &= \frac{1}{2}A^{\alpha\beta}_{\sigma\tau}\nabla_{\boldsymbol{x}}\int d^3\boldsymbol{p}d^3\boldsymbol{q}\ f(\mathbf{x},\mathbf{p};t)^{\sigma}_{\alpha}f(\mathbf{x},\mathbf{q};t)^{\tau}_{\beta} - \\
&\quad -\frac{1}{2}\sum_i \partial_i \int d^3\boldsymbol{p}d^3\boldsymbol{q}\ \boldsymbol{p}\ f(\mathbf{x},\mathbf{p};t)^{\sigma}_{\alpha}\boldsymbol{A}_i{}^{\alpha\beta}_{\sigma\tau}\cdot\nabla_{\boldsymbol{x}}f(\mathbf{x},\mathbf{p};t)^{\tau}_{\beta}.
\end{aligned}
$$

$$
\frac{d}{dt}\boldsymbol{s}(\boldsymbol{x},t) = \frac{1}{2}\boldsymbol{\sigma}^{\alpha}_{\beta}\left\{\frac{\hbar}{2}\sum_i\partial_i\int d^3\boldsymbol{p}d^3\boldsymbol{q}\left[\boldsymbol{A}_i{}^{\beta\sigma}_{\tau\tau'}f(\mathbf{x},\mathbf{p};t)^{\tau}_{\alpha} - \boldsymbol{A}_i{}^{\tau\sigma}_{\alpha\tau'}f(\mathbf{x},\mathbf{p};t)^{\beta}_{\tau}\right]\cdot\nabla_{\boldsymbol{x}}f(\mathbf{x},\mathbf{q};t)^{\tau'}_{\sigma}\right\}
$$

Using the approximations of short range interparticle potential, which is no more than linearly dependent on momentum, of short variation of the Boltzmann quasi-distribution over the range of the potential and over the Compton wavelength.

SUMMARY AND OUTLOOK

- We aim at obtaining a general formulation of relativistic hydrodynamics with spin (possible application to QGP?).

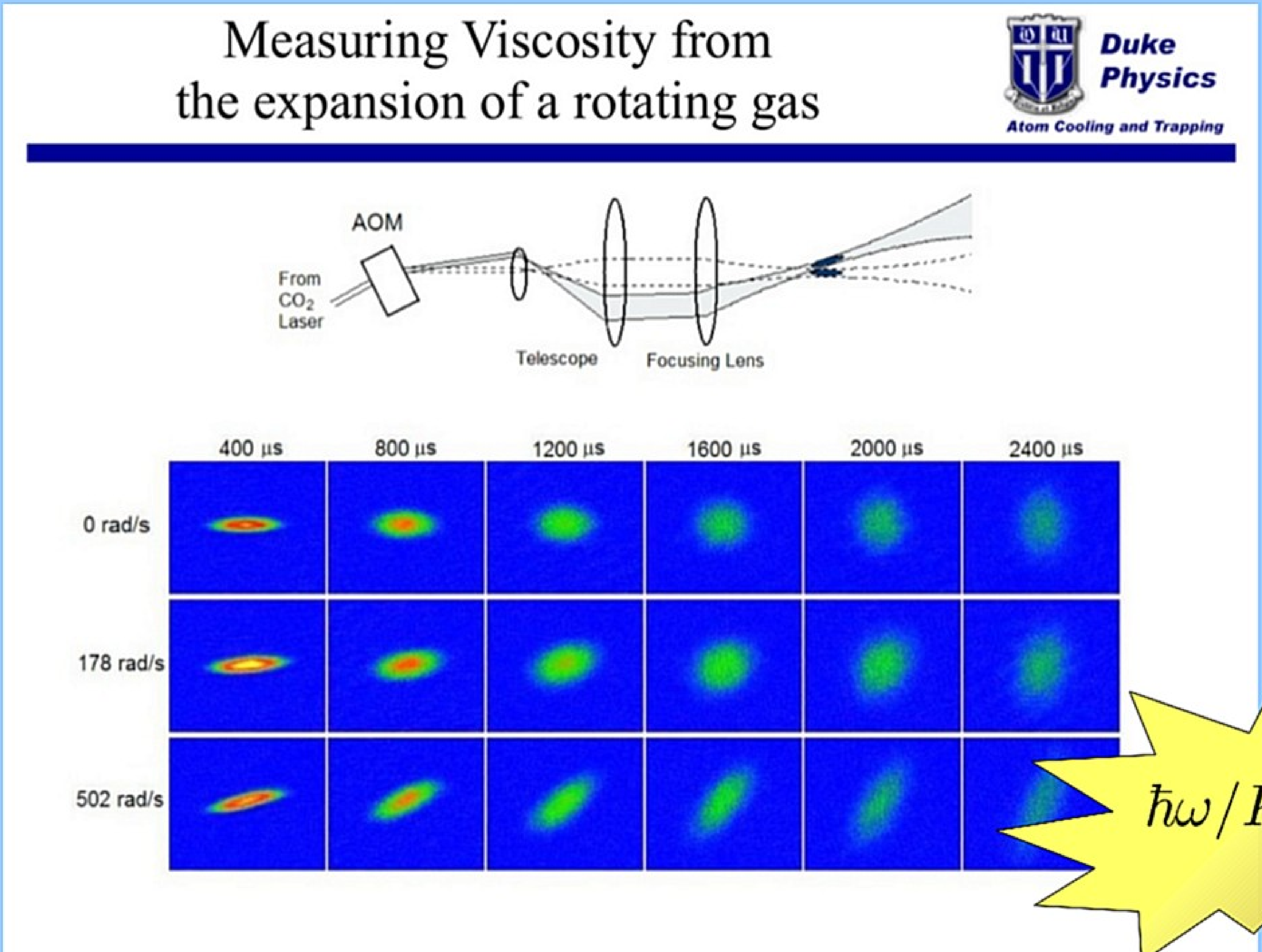
- Microscopic stress-energy and spin tensor that give the same total momentum and angular momentum are not macroscopically equivalent. Which is the right couple?

- Can it be tested experimentally?

- Kinetic approach: new generation of transport coefficients related to spin. So far, non relativistic treatment, hope to extending it to relativistic case.

Can we measure the polarization effects?

J. E. Thomas, Nucl. Phys. A830, 665C (2009).



$$\hbar\omega/KT \sim 0.1$$