Hagedorn’s Model

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Bibliography

In high-energy particle collisions (p,e) many hadrons were created with larger and larger invariant mass. Theorists were at work to find a consistent description and classification of hadrons. Two possible ways:

- Quark Model (→ QCD, middle '70)
- Hagedorn’s Model (1965), based on statistical properties

### Hagedorn’s approach

Self-similar scheme for the composition and decay of hadrons and their resonances (“fireballs”) (Statistical Bootstrap Model)

A heavy particle is a resonant state formed by lighter particles, in a self similarity pattern:

> a fireball consists of fireballs, which in turn consist of fireballs, and so on...\(^1\)

System of non-interacting particles, in which the formation and decay of resonances simulates the interaction.

\(^1\)R. Hagedorn: “Statistical thermodynamics of strong interactions at high energies”, Nuovo Cim. Suppl. 3, 147 (1965)
Self-similarity in mathematics

- Sierpinski Triangle (1915)


- Number theory: how many ways are there of decomposing an integer into a sum of integers?

  \[
  \begin{align*}
  1 &= 1 \\
  2 &= 2, 1+1 \\
  3 &= 3, 2+1, 1+2, 1+1+1 \\
  4 &= 4, 3+1, 2+2, 2+1+1, 1+2+1, 1+1+2, 1+1+1+1 \\
  \end{align*}
  \]

  There are \( p(n) = 2^{n-1} = \frac{1}{2} e^{n \ln 2} \) ways of partitioning an integer \( n \) into ordered partitions: \( p(n) \) grows exponentially in \( n \).

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The Statistical Bootstrap Model

Let us consider a system of non-interacting particles with momentum \( \vec{p}_\alpha \), mass \( m_\gamma \), energy \( \varepsilon_{\alpha\gamma} = \sqrt{\vec{p}_\alpha^2 + m_\gamma^2} \); let \( \nu_{\alpha\gamma} \) be their multiplicity.

The total energy of the system is:

\[
E = \sum_{\alpha\gamma} \nu_{\alpha\gamma} \varepsilon_{\alpha\gamma}
\]

The Grand-Partition Function (Grand-canonical description with \( \mu = 0 \)) is:

\[
\mathcal{Z}(V, T) = \sum_{\{\nu\}} \exp \left\{ -\frac{1}{T} \sum_{\alpha\gamma} \nu_{\alpha\gamma} \varepsilon_{\alpha\gamma} \right\}
\]

or, in the continuum limit,

\[
\mathcal{Z}(V, T) = \int_0^\infty \sigma(E, V) \exp \left\{ -\frac{E}{T} \right\} dE
\]
Short-hand notation: \( x_{\alpha \gamma} \equiv \exp \left\{ -\frac{\varepsilon_{\alpha \gamma}}{T} \right\} \):

\[
Z(V, T) = \sum_{\nu} \exp \left\{ -\frac{1}{T} \sum_{\alpha \gamma} \nu_{\alpha \gamma} \varepsilon_{\alpha \gamma} \right\} = \sum_{\nu} \prod_{\alpha \gamma} x_{\alpha \gamma}^{\nu_{\alpha \gamma}} = \prod_{\alpha \gamma} \left[ \sum_{\nu} x_{\alpha \gamma}^{\nu_{\alpha \gamma}} \right] \]

with the occupation numbers: \( \nu_{\alpha \gamma} \mapsto \begin{cases} \nu_{\alpha \beta} = 0, 1, 2, \ldots \quad \text{Bosons} \\ \nu_{\alpha \phi} = 0, 1 \quad \text{Fermions} \end{cases} \)

\[
Z(V, T) = \prod_{\alpha \phi} (1 + x_{\alpha \phi}) \prod_{\alpha \beta} \frac{1}{1 - x_{\alpha \beta}} \\
\log Z(V, T) = \sum_{\alpha \phi} \log(1 + x_{\alpha \phi}) - \sum_{\alpha \beta} \log(1 - x_{\alpha \beta})
\]

\[
\log Z(V, T) = \frac{V}{2\pi^2} \int_{0}^{\infty} p^2 dp \left[ \int_{0}^{\infty} \rho_F(m) \log(1 + x_{p,m}) dm - \int_{0}^{\infty} \rho_B(m) \log(1 - x_{p,m}) dm \right]
\]

\[
\log(1 \pm x) = \pm x - x^2 \pm x^3 - \ldots \quad (x \equiv \exp\left\{ -\frac{\varepsilon}{T} \right\} \Rightarrow 0 < x < 1)
\]

\[
\log Z(V, T) = \frac{V}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \int_{0}^{\infty} p^2 \rho(m; n) x_{p,m}^n dp dm
\]

where

\[
x_{p,m}^n \equiv \exp \left\{ -\frac{n}{T} \sqrt{p^2 + m^2} \right\} \quad \rho(m; n) = \rho_B(m) - (-1)^n \rho_F(m)
\]

By integrating over momenta (\( K_2 \)):

\[
Z(V, T) = \exp \left\{ \frac{VT}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{m_0}^{\infty} \rho(m; n) m^2 K_2 \left( \frac{nm}{T} \right) dm \right\}
\]
We have obtained two expressions for $Z$:

\[
Z(V, T) = \int_0^\infty \sigma(E, V) \exp\left\{ -\frac{E}{T} \right\} \, dE
\]

\[
= \exp\left\{ \frac{VT}{2\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_{m_0}^\infty \rho(m; n) m^2 K_2 \left( \frac{nm}{T} \right) \, dm \right\}
\]

Hagedorn imposed the *logarithm bootstrap condition* ("weak condition")

\[
\frac{\log \rho(m; n)}{\log \sigma(m, V_0)} \xrightarrow[m \to \infty]{} 1
\]

i.e. asymptotic equality of entropies. 
$\sigma$ and $\rho$ differ by som algebraic factor in $m$: $\sigma(E)$ counts all the states of the system in $V_0$, including, for instance, those whit very large angular momentum (=collective motion) which are not "fireballs" and therefore are not counted in $\rho$.

Hagedorn solved this equation by iterations:

\[
\rho^{(0)} \rightarrow Z^{(0)} \rightarrow \sigma^{(0)}
\]

\[
\log \sigma^{(0)} \equiv \log \rho^{(1)}
\]

\[
\rho^{(1)} \rightarrow Z^{(1)} \rightarrow \sigma^{(1)}
\]

\[
\log \sigma^{(1)} \equiv \log \rho^{(2)}
\]

\[
\ldots
\]

Starting from a simple $\rho^{(0)}(m)$, in a few iterations one gets an exponential behaviour:

\[
\rho(m) = Am^a e^{m/T_H} \quad \quad \sigma(m) = Bm^b e^{m/T_H}
\]

The logarithm bootstrap condition is satisfied:

\[
\frac{\log \rho(m; n)}{\log \sigma(m, V_0)} = \frac{m/T_H + a \log m + \log A}{m/T_H + b \log m + \log B} \xrightarrow[m \to \infty]{} 1
\]
Hagedorn’s solution: $a = -\frac{5}{2}$

$$\rho_H(m) = A m^{-5/2} e^{m/T_H}$$

with $T_H \sim 150 - 180$ MeV.

A few years later, Nahm$^2$ solved the equation analytically (with a “strong” condition, conservation laws), and found $a = -3$:

$$\rho(m) = A m^{-3} e^{m/T_H}$$

with:

$$\frac{V_0 T_H^3}{2 \pi^2} \left( \frac{m_0}{T_H} \right)^2 K_2 \left( \frac{m_0}{T_H} \right) = 2 \log 2 - 1$$

For $m_0 = m_\pi$: $T_H \simeq 150$ MeV.

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Experimental estimate of $T_H$


Finally we discuss the transverse momentum distribution of secondaries produced in P–P collisions of fixed energies. In 1961, Cozoni, Koester and Perkins pointed out that the distribution function

$$dN/dp_L \propto p_L \exp(-a p_L)$$

(4) fits the 10 to 30 GeV pion production data from CERN and Brookhaven as well as cosmic ray data up to $10^9$ GeV [15]. We note that eqs. (1) or (2) yields the same transverse momentum distribution function for elastically scattered protons as long as we keep away from angles near 90°. The plots are consistent with eq. (4) and a value of $1/a = 160$ MeV/c.

No firm theoretical explanation has yet been given of why a simple exponential, $\exp(-a p)$, should appear to dominate high energy physics. Recent

Hagedorn calculated the transverse momentum distribution in its model and found a natural explanation!
A closer look...

Let us study the high-temperature limit of the Hagedorn’s partition function (only \( n = 1 \)):

\[
\log Z(T, V) \simeq \frac{VT}{2\pi^2} \int_{m_0}^{\infty} m^2 \rho(m, 1) K_2 \left( \frac{m}{T} \right) \, dm
\]

\[
\propto \frac{VT}{2\pi^2} \int_{m_0}^{\infty} m^2 m^a e^{m/T_H} K_2 \left( \frac{m}{T} \right) \, dm
\]

for \( z \to \infty \): \( K_2(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} + O(z^{-2}) \)

\[
e^{m/T_H} K_2 \left( \frac{m}{T} \right) \sim e^{-m \left( \frac{1}{T} - \frac{1}{T_H} \right)}
\]

\( Z \) diverges exponentially if \( T > T_H \)!

Hagedorn: \( T_H \) is the limiting temperature for hadronic matter

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A new phase


**EXPONENTIAL HADRONIC SPECTRUM AND QUARK LIBERATION**

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The exponentially increasing spectrum proposed by Hagedorn is not necessarily connected with a limiting temperature, but it is present in any system which undergoes a second order phase transition. We suggest that the “observed” exponential spectrum is connected to the existence of a different phase of the vacuum in which quarks are not confined.

This is the current interpretation of \( T_H \)
Conclusions

- Hagedorn’s Model was proposed and studied between 1965 and 1975. It was abandoned in favor of the QCD.
- It was recovered in the middle of ’90, when it was observed that the total multiplicity of hadronic particles produced in high energy collisions ($e^+ - e^-$, $p - p$, ...) could be accurately described by thermal models: $N \propto e^{m/T}$, with $T \sim 150$ MeV.
- At present, its modern version (the Hadron Resonance Gas Model) is widely used to study the hadronic phase in the heavy-ion collisions.