Written Exam CHI 0026 Quantum Mechanics

17 February 2015, 2.00–4.00 PM

Please read the following IMPORTANT INSTRUCTIONS

Answer at most TWO questions. Only two questions will be marked.

You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

Answer at most TWO questions

1. The Hamiltonian for a one-dimensional simple harmonic oscillator of mass m and angular frequency ω is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$

a) Show that, if $[x, p] = i\hbar$, and $< x(t) >$ and $< p(t) >$ are the expectation values of $x(t)$ and $p(t)$ at time t, they satisfy *precisely* the classical equations, i.e.

$$m\frac{d < x(t) >}{dt} = < p(t) >, \quad \frac{d < p(t) >}{dt} = -m\omega^2 < x(t) > .$$

b) The state of the oscillator at time $t = 0$ is given by

$$\psi(x) = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} (\frac{\alpha x}{\sqrt{2}} + (\alpha x)^2) \exp{(-\frac{\alpha^2 x^2}{2})}, \quad \alpha = (\frac{m\omega}{\hbar})^{\frac{1}{2}}.$$

and the three lowest energy wavefunctions are

$$\psi_0(x) = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} \exp{(-\frac{\alpha^2 x^2}{2})},$$

$$\psi_1(x) = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} (\sqrt{2}\alpha x) \exp{(-\frac{\alpha^2 x^2}{2})},$$

$$\psi_2(x) = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} \frac{(2\alpha^2 x^2 - 1)}{\sqrt{2}} \exp{(-\frac{\alpha^2 x^2}{2})}.$$

What are their corresponding energies E_0, E_1, E_2?

c) Show that the initial state can be expressed as

$$\psi(x) = A_0\psi_0(x) + A_1\psi_1(x) + A_2\psi_2(x)$$

with A_0, A_1, A_2 constants, and evaluate these constants. What is their meaning?

(CONTINUED)

2. Assume that, in spherical polar coordinates (r, θ, ϕ), the z-component of angular momentum L_z is represented by $-i\hbar\partial/\partial\phi$, where ϕ is the polar angle, in the range $(0, 2\pi)$.

a) Show that L_z has eigenvalues $m\hbar$, with m an integer.

b) The following three states describe the motion of a particle moving in three dimensions

$$\psi_1(r) = \frac{c}{\sqrt{2}}(x + iy) \exp -r^2$$

$$\psi_2(r) = \frac{c}{\sqrt{2}}(x - iy) \exp -r^2$$

$$\psi_3(r) = cz \exp -r^2$$

where c is a normalisation constant and $r^2 = x^2 + y^2 + z^2$.

Verify, using Cartesian coordinates x, y, z, that each of these states is an eigenstate of L_z and find the corresponding eigenvalues.

c) Describe the possible outcomes of measuring L_z for a particle in the state

$$\psi(r) = \frac{c}{\sqrt{5}}(2z - x) \exp -r^2$$

3. A proton of mass m is confined in an infinite square well with walls at $x = 0$ and $x = L$.

a) Sketch the first three normalised energy eigenstates ψ_n, $n = 1, 2, 3$ and calculate their energies.

b) Determine the probability $P_n(\frac{1}{a})$ that the proton is between $x = 0$ and $x = \frac{L}{a}$ in the state ψ_n.

c) What is $P_n(\frac{1}{a})$ for n large? How does this compare with the classical result?

4.

a) Define a hermitian operator in Quantum Mechanics

b) Show that the eigenvalues of a hermitian operator are real, and give two examples.

c) Define parity in Quantum Mechanics. Is the corresponding operator hermitian?

d) Show that wave functions of opposite parity are orthogonal, and give an example.

e) Show that the wavefunctions of the infinite potential well ($V = 0$ if $|x| < a$, $V = \infty$ otherwise), have definite parity.