Written Exam CHI 0026 Quantum Mechanics

23 February 2016, 2.00–4.00 PM

Please read the following IMPORTANT INSTRUCTIONS

Answer at most TWO questions. Only two questions will be marked.

You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1.

(a) Assume that, in spherical polar coordinates (r, θ, ϕ), the z-component of angular momentum L_z is represented by $-i\hbar\partial/\partial\phi$, where ϕ is the polar angle, in the range $(0, 2\pi)$. Show that L_z has eigenvalues $m\hbar$, with m an integer.

(b) The following three states describe the motion of a particle moving in three dimensions

$$\psi_1(r) = \frac{c}{\sqrt{2}}(x + iy) \exp -r^2$$

$$\psi_2(r) = \frac{c}{\sqrt{2}}(x - iy) \exp -r^2$$

$$\psi_3(r) = cz \exp -r^2$$

where c is a normalisation constant and $r^2 = x^2 + y^2 + z^2$. Verify, using Cartesian coordinates x, y, z, that each of these states is an eigenstate of L_z and find the corresponding eigenvalues.

(c) Describe the possible outcomes of measuring L_z for a particle in the state

$$\psi(r) = \frac{c}{\sqrt{5}}(2z - x) \exp -r^2$$

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2.

(a) If O is a quantum–mechanical operator, what is the definition of the corresponding Hermitian conjugate operator $O^\dagger$?

(b) Define what is meant by a Hermitian operator in quantum mechanics

(c) Show that d/dx is not a Hermitian operator. What is its Hermitian conjugate, $(d/dx)^\dagger$?

(d) Prove that, for any two operators A and B, $(AB)^\dagger = B^\dagger A^\dagger$.

(CONTINUED)

3.

The wave function $\psi(x) = A\; x\; \mathrm{e}^{-\alpha x^2}$, with A, α constants, describes an energy eigenstate of a one–dimensional harmonic oscillator of mass m and angular frequency ω.

(a) Is $\psi(x)$ an eigenstate of parity?

(b) What is α in terms of m and ω?

(c) Which state is this? The ground state, the first excited state, the second excited state etc?

(d) Normalise the wave function $\psi(x)$, i.e. determine A in terms of α, and calculate the expectation values $< x >$ and $< p >$ of position and momentum.

(*You may need* $\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\lambda\, x^2} = \sqrt{\frac{\pi}{\lambda}}, \quad \int_{-\infty}^{\infty} dx\, x^2\, \mathrm{e}^{-\lambda\, x^2} = \frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}$)

4.

(a) Verify the following commutation relations for operators A, B, and C.

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C$$

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(b) Consider the Hamiltonian

$$H = \frac{p^2}{2m} + Kx^4$$

with m and K constants. Assuming the commutation rule

$$[x, p] = i\hbar$$

show that

$$\frac{i}{\hbar}[H, x] = \frac{p}{m}$$

and calculate the commutator

$$\frac{i}{\hbar}[H, p]$$

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