Written Exam CHI 0026 Quantum Mechanics

2 February 2016, 2.00–4.00 PM

Please read the following IMPORTANT INSTRUCTIONS

Answer at most TWO questions. Only two questions will be marked.

You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1. A proton of mass m is confined in an infinite square well with walls at $x = 0$ and $x = L$.

1. Sketch the first three normalised energy eigenstates ψ_n, $n = 1, 2, 3$ and calculate their energies.
2. If the proton is in the state ψ_n, calculate the probability P_n that it is confined to the centre of the well, i.e. between $L/4$ and $3L/4$, showing how P_n depends on n.
3. Compare your P_n, for n large, with the classical probability $P(x)dx = dx/L$
4. What is P_1?
5. What are the energy and wavelength of the photon that is emitted when the proton jumps down from the first excited state to the ground state?

N.B.You might need $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

2. At time $t = 0$ a quantum oscillator of frequency ω and mass m is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \beta|1\rangle\right)$$

where $|0\rangle$ and $|1\rangle$ are normalised energy eigenstates, with energies E_0 and E_1 respectively, and β is a constant.

1. What condition must β satisfy such that $|\psi\rangle$ is correctly normalised?
2. Is $|\psi\rangle$ stationary, i.e. is it an energy eigenstate? Justify your answer.
3. What is the state $|\psi(t)\rangle$ at a later time t?
4. Calculate the expectation value of energy $\langle E\rangle$ in the state $|\psi(t)\rangle$ i.e. at a later time t.

3. In Quantum Mechanics,

1. which operators can be associated to observable physical quantities, and why? Give two examples.

2. what are the possible outcomes of measuring an observable associated to an operator A? How do you express the probabilities of these outcomes?

3. are the functions

   (a) $\psi(x) = \mathrm{e}^{-\frac{1}{2}\alpha\, x^2}$

   (b) $\psi(x) = \mathrm{e}^{i(kx-\omega t)}$

   eigenfunctions of the momentum operator p? If so, what are their eigenvalues?

4. is the function $\psi(x) = x\,\mathrm{e}^{-\frac{1}{2}\alpha\, x^2}$ an eigenfunction of the operator $p^2+\beta x^2$? If so, for what value of β, in terms of α and $\hbar$? For which eigenvalue?

4. Consider a single particle with angular momentum $\vec{L} = (L_x, L_y, L_z)$

1. Define the operators L_x, L_y, L_z in terms of the position and momentum operators for the particle.

2. Assuming $[x, p_x] = i\hbar$ etc, show that

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y$$

3. Defining $L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2$, show that

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0,$$

4. Show that the eigenvalues of L^2 are all positive or zero.

5. A point particle moving in three dimensions is described by a wave function

$$\phi(\vec{r}) = cz\exp\left(-\alpha r^2\right)$$

   where c and α are constants and $r^2 = x^2 + y^2 + z^2$. Assuming that $\phi(\vec{r})$ is an eigenfunction of L^2, is it also an eigenfunction of L_z? If so, what is its eigenvalue?