Written Exam MFN 0740 Quantum Mechanics

22 February 2011, 3.15–5.15 PM

Please read the following INSTRUCTIONS

A. Answer at most TWO questions. A pass is obtained for one complete answer, and full marks for two complete answers.
B. You may not use notes or textbooks, but the lecture notes are available for consultation at the front desk.

1.a) Verify the following commutation relations for operators A, B, and C.

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C$$

b) The Hamiltonian for a one-dimensional simple harmonic oscillator of mass m and angular frequency ω is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

Assuming the commutation rule

$$[x, p] = i\hbar$$

show that

$$\frac{i}{\hbar}[H, x] = \frac{p}{m}$$

and

$$\frac{i}{\hbar}[H, p] = -m\omega^2 x$$

2. At $t = 0$ a stream of particles arriving from $x = -\infty$ of mass m and energy E encounter a potential step of height W (with $W < E$) at $x = 0$. Assume the wavefunction for $x < 0$ is

$$\psi(x) = \mathrm{e}^{ikx} + R\mathrm{e}^{-ikx} \ .$$

a) What is k?

b) What is the *form* of the wavefunction for $x > 0$?

c) calculate the reflection coefficient R.

(continued)

3. The wavefunction of a free particle of mass m, moving in one dimension, at time $t = 0$, is given by

$$\psi(x) = \mathcal{N} \mathrm{e}^{-\frac{1}{2}\alpha\, x^2} \ .$$

where $\mathcal{N}$ and α are positive constants.

a) Evaluate $\mathcal{N}$, in terms of α, by requiring that ψ be normalised.

b) Calculate the dispersions (standard deviations) Δx and Δp, for the state ψ, and verify that they satisfy the Heisenberg principle.

4. a) In Quantum Mechanics, which operators can be associated to observable physical quantities, and why? Give at least one example.

b) Are the functions

i)

$$\psi(x) = \mathrm{e}^{-\frac{1}{2}\alpha\, x^2} \ .$$

ii)

$$\psi(x) = \mathrm{e}^{i(kx-\omega t)} \ .$$

eigenfunctions of the momentum operator? If so, what are their eigenvalues?

c) What are the possible outcomes of measuring an observable associated to an operator A? How do you express the probabilities of these outcomes?

Useful formulae

$$\int_{-\infty}^{\infty} dx\, x^2\, \mathrm{e}^{-\alpha\, x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} \ ,$$

$$\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\alpha\, x^2} = \sqrt{\frac{\pi}{\alpha}}$$