Written Exam MFN 0740 Quantum Mechanics

30 January 2012, 3.00–5.00 PM

Please read the following INSTRUCTIONS

A. Answer at most TWO questions. A pass is obtained for one complete answer.
B. You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1

a) Assume that, in spherical polar coordinates (r, θ, ϕ), the z-component of angular momentum L_z is represented by $-i\hbar\partial/\partial\phi$, where ϕ is the polar angle, in the range $(0, 2\pi)$. Show that L_z has eigenvalues $m\hbar$, with m an integer.

b) The following three states describe the motion of a particle moving in three dimensions

$$\psi_1(r) = \frac{c}{\sqrt{2}}(x + iy)\exp -r^2$$

$$\psi_2(r) = \frac{c}{\sqrt{2}}(x - iy)\exp -r^2$$

$$\psi_3(r) = cz\exp -r^2$$

where c is a normalisation constant and $r^2 = x^2 + y^2 + z^2$.

Verify, using Cartesian coordinates x, y, z, that each of these states is an eigenstate of L_z and find the corresponding eigenvalues.

c) Describe the possible outcomes of measuring L_z for a particle in the state

$$\psi(r) = \frac{c}{\sqrt{5}}(2z - x)\exp -r^2$$

2

a) If O is a quantum–mechanical operator, what is the definition of the corresponding Hermitian conjugate operator $O^\dagger$?

b) Define what is meant by a Hermitian operator in quantum mechanics

c) Show that d/dx is not a Hermitian operator. What is its Hermitian conjugate, $(d/dx)^\dagger$?

d) Prove that, for any two operators A and B, $(AB)^\dagger = B^\dagger A^\dagger$.

3

A particle is in an infinitely deep potential well, where $V = 0$ for $|x| \leq a$ and $V = \infty$ for $|x| > a$. Its wave function is $\psi(x) = A \sin kx + B \cos kx$ for $|x| \leq a$, $\psi(x) = 0$ for $|x| > a$, with A, B and k constants.

a) Show that $k = n\pi/(2a)$, $n = 1, 2, 3....$ with $A = 0$ for n odd, $B = 0$ for n even.

b) Show that $\psi(x)$ is an eigenfunction of the Hamiltonian operator, and calculate its energy.

c) Consider the same problem with a shift of origin, so that the well now runs from $x = 0$ to $x = 2a$. What is the form of the wavefunction in this case?

d) The probability density for finding the particle at a given value of x is $|\psi(x)|^2$. Use this fact to *normalise* the wavefunction, i.e. calculate the constants A and B.

e) Calculate the expectation values of position $\langle x \rangle$ and momentum $\langle p \rangle$. *(N.B. By simple reasoning you can deduce the result..)*

4

Prove the following results for an eigenstate of the Hamiltonian of a one-dimensional simple harmonic oscillator

a) The expectation values of the position and momentum are zero

b) The expectation values of the potential and kinetic energies are equal

c) The uncertainties in position and momentum Δx and Δp, satisfy the relation $\Delta x \Delta p = (n + \frac{1}{2})\hbar$, where n is the quantum number of the state.

d) The ground state eigenfunction is a Gaussian, and the parity of the state n is even for n even, and odd for n odd.

(You may need

$$p = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right) \;\; , \;\; x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

where $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.*)*