Written Exam MFN 1072 Quantum Mechanics

28 January 2014, 2.00–4.00 PM

Please read the following INSTRUCTIONS

Answer at most TWO questions. A pass is obtained for one complete answer. You may not use notes or textbooks. but the lecture notes etc are available for consultation at the front desk.

Answer at most TWO questions

1. A particle in a potential well is initially in a state whose wavefunction $\psi(x,0)$ is an equal-weight superposition of the ground state and first excited state wavefunctions: $\psi(x,0) = C(\psi_1(x) + \psi_2(x))$.

a) Show that the value $C = \frac{1}{\sqrt{2}}$ normalises $\psi(x,0)$, assuming that ψ_1 and ψ_2 are themselves normalised.

b) Assuming you know the ground and first excited state energies E_1 and E_2, determine $\psi(x,t)$ at any later time t.

c) Show that the average energy $< E >$ for $\psi(x,t)$ (i.e. at time t) is the arithmetic mean of E_1 and E_2, that is

$$< E >= \frac{E_1 + E_2}{2}.$$

d) Determine the uncertainty ΔE of energy for $\psi(x,t)$.

2.

a) In Quantum Mechanics, which operators can be associated to observable physical quantities, and why? Give at least one example.

b) What are the possible outcomes of measuring an observable associated to an operator A? How do you express the probabilities of these outcomes?

c) Are the functions

i) $\psi(x) = \mathrm{e}^{-\frac{1}{2}\alpha\, x^2}$
ii)$\psi(x) = \mathrm{e}^{i(kx-\omega t)}$

eigenfunctions of the momentum operator p? If so, what are their eigenvalues?

iii) For what value of β is the function in i) i.e. $\psi(x) = \mathrm{e}^{-\frac{1}{2}\alpha\, x^2}$ an eigenfunction of the operator $p^2 + \beta x^2$? For what eigenvalue? Express your answers in terms of α and $\hbar$.

(CONTINUED)

3.

a) Define the single particle angular momentum operators L_x, L_y, L_z in terms of the position and momentum operators x and p for the particle.

b) Assuming $[x, p_x] = i\hbar$ etc, show that

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y$$

c) Defining $L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2$, show that

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0,$$

d) Show that the eigenvalues of L^2 are all positive or zero.

e) A point particle moving in three dimensions is described by a wave function

$$\phi(\vec{r}) = cz \exp\left(-\alpha r^2\right)$$

where c and α are constants and $r^2 = x^2 + y^2 + z^2$. Assuming that $\phi(\vec{r})$ is an eigenfunction of L^2, is it also an eigenfunction of L_z? If so, what is its eigenvalue?

4. Prove the following results for an eigenstate of the Hamiltonian of a one - dimensional simple harmonic oscillator

a) The expectation values of the position and momentum are zero

b) The expectation values of the potential and kinetic energies are equal

c) The uncertainties in position and momentum Δx and Δp, satisfy the relation $\Delta x \Delta p = (n + \frac{1}{2})\hbar$, where n is the quantum number of the state.

N.B.You may need

$$p = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right) \;,\; x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

where $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.