Written Exam CHI 0026 Quantum Mechanics

27 January 2015, 2.00–4.00 PM

Please read the following IMPORTANT INSTRUCTIONS

Answer at most TWO questions. Only two questions will be marked.

You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1. i) In Quantum Mechanics, which operators can be associated to observable physical quantities, and why? Give at least one example.

ii) What are the possible outcomes of measuring an observable associated to an operator? How do you express the probabilities of these outcomes?

iii) Are the functions

a) $\psi_1(x) = \sin kx$

b) $\psi_2(x) = \sin kx - \cos kx$

c) $\psi_3(x) = \cos kx + i \sin kx$

d) $\psi_4(x) = \mathrm{e}^{-\alpha x^2}$

eigenfunctions of the momentum operator? If so, what are their eigenvalues?

2. A particle of mass m in one dimension x is subject to a potential $U(x)$. At time t its wave function is

$$\psi(x,t) = A\ \mathrm{e}^{-\dfrac{\alpha x^2}{2} - i\beta t}$$

where A, α, β are constants, and α, β are real.

a) Evaluate A, by requiring that ψ be normalised.

b) Calculate the potential $U(x)$.

c) Calculate the particle's total energy.

d) Calculate the dispersions (standard deviations) Δx and Δp, for the state $\psi(x,t)$, and verify that they satisfy the Heisenberg principle.

You may need

$$\int_{-\infty}^{\infty} dx\, x^2\, \mathrm{e}^{-\alpha\, x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}, \qquad \int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\alpha\, x^2} = \sqrt{\frac{\pi}{\alpha}}.$$

3. Prove the following results for an eigenstate of the Hamiltonian of a one-dimensional simple harmonic oscillator

a) The expectation values of the position and momentum are zero

b) The expectation values of the potential and kinetic energies are equal

c) The uncertainties in position and momentum Δx and Δp, satisfy the relation $\Delta x \Delta p = (n + \frac{1}{2})\hbar$, where n is the quantum number of the state.

d) The ground state eigenfunction is a Gaussian, and the parity of the state n is even for n even, and odd for n odd.

You may need

$$p = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right) \ , \qquad x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

where $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

4. a) Define the single particle angular momentum operators L_x, L_y, L_z in terms of the position and momentum operators for the particle.

b) Assuming $[x, p_x] = i\hbar$ etc, show that

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y$$

c) Defining $L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2$, show that

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0,$$

d) Show that the eigenvalues of L^2 are all positive or zero.

e) A point particle moving in three dimensions is described by a wave function

$$\phi(\vec{r}) = cz\exp\left(-\alpha r^2\right)$$

where c and α are constants and $r^2 = x^2 + y^2 + z^2$. Assuming that $\phi(\vec{r})$ is an eigenfunction of L^2, is it also an eigenfunction of L_z? If so, what is its eigenvalue?