# Written Exam MFN 1072 Quantum Mechanics

**1 July 2014, 2.30–4.30 PM**

**Please read the following IMPORTANT INSTRUCTIONS**

**Answer at most TWO questions. Only two questions will be marked. A pass is obtained for one complete answer.**

**You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.**

**1.**

The wavefunction of a free particle of mass $m$, moving in one dimension, at time $t = 0$, is given by

$$\psi(x) = \mathcal{N} \mathrm{e}^{-\frac{1}{2}\alpha\, x^2} \; .$$

where $\mathcal{N}$ and $\alpha$ are positive constants.

a) Evaluate $\mathcal{N}$, in terms of $\alpha$, by requiring that $\psi$ be normalised.

b) Calculate the dispersions (standard deviations) $\Delta x$ and $\Delta p$, for the state $\psi$, and verify that they satisfy the Heisenberg principle.

**N.B.You may need**

$$\int_{-\infty}^{\infty} dx\, x^2\, \mathrm{e}^{-\lambda\, x^2} = \frac{1}{2\lambda}\sqrt{\frac{\pi}{\lambda}}, \qquad \int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\lambda\, x^2} = \sqrt{\frac{\pi}{\lambda}}.$$

**2.**

a) Define a hermitian operator in Quantum Mechanics

b) Show that the eigenvalues of a hermitian operator are real, and give two examples.

c) Define parity in Quantum Mechanics. Is the corresponding operator hermitian?

d) Show that wave functions of opposite parity are orthogonal, and give an example.

e) Show that the wavefunctions of the infinite potential well ($V = 0$ if $|x| < a$, $V = \infty$ otherwise), have definite parity.

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**3.**

A particle in a potential well $U(x)$ is initially in a state whose wavefunction $\psi(x,0)$ is an equal-weight superposition of the ground state and first excited state wavefunctions:

$$\psi(x,0) = C(\psi_1(x) + \psi_2(x))$$

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a) Show that the value $C = \frac{1}{\sqrt{2}}$ normalises $\psi(x,0)$, assuming that $\psi_1$ and $\psi_2$ are themselves normalised.

b) Assuming you know the ground and first excited state energies $E_1$ and $E_2$, determine $\psi(x,t)$ at any later time t.

c) Show that the average energy $< E >$ for $\psi(x,t)$ is the arithmetic mean of $E_1$ and $E_2$, that is

$$< E > = \frac{E_1 + E_2}{2}.$$

d) Determine the uncertainty $\Delta E$ of energy for $\psi(x,t)$.

**4.**

Assume that, in spherical polar coordinates $(r, \theta, \phi)$, the $z$-component of angular momentum $L_z$ is represented by $-i\hbar\partial/\partial\phi$, where $\phi$ is the polar angle, $\phi \,\epsilon\, [0, 2\pi]$.

a) Show that $L_z$ has eigenvalues $m\hbar$, with $m$ an integer.

b) The following three states describe the motion of a particle moving in three dimensions

$$\psi_1(r) = \frac{c}{\sqrt{2}}(x + iy)\exp -r^2$$

$$\psi_2(r) = \frac{c}{\sqrt{2}}(x - iy)\exp -r^2$$

$$\psi_3(r) = cz\exp -r^2$$

where $c$ is a normalisation constant and $r^2 = x^2 + y^2 + z^2$.

Verify, using Cartesian coordinates $x, y, z$, that each of these states is an eigenstate of $L_z$ and find the corresponding eigenvalues.

c) Describe the possible outcomes of measuring $L_z$ for a particle in the state

$$\psi(r) = \frac{c}{\sqrt{5}}(2z - x)\exp -r^2$$

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