Written Exam CHI 0026 Quantum Mechanics

19 July 2016, 2.00–4.00 PM

Please read the following IMPORTANT INSTRUCTIONS

Answer at most TWO questions. Only two questions will be marked. A pass is obtained for one complete answer.

You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1. A proton of mass m is confined in an infinite square well with walls at $x = 0$ and $x = L$.

1. Sketch the first three normalised energy eigenstates ψ_n, $n = 1, 2, 3$ and calculate their energies.
2. Calculate the probability P_n that the proton is confined to the interval $[0, d]$ where $0 \leq d \leq L$, showing how P_n depends on n and d.
3. What is P_n, for n large?
4. What are the energy and wavelength of the photon that is emitted when the proton jumps down from the first excited state to the ground state?

N.B.You might need $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

2. At time $t = 0$ a quantum oscillator of frequency ω and mass m is in the state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \beta|1\rangle\right)$$

where $|0\rangle$ and $|1\rangle$ are energy eigenstates, with energies E_0 and E_1 respectively.

1. What condition must β satisfy such that $|\psi\rangle$ is correctly normalised?
2. Is $|\psi\rangle$ stationary, i.e. is it an energy eigenstate? Justify your answer.
3. What is the state $|\psi(t)\rangle$ at a later time t?
4. Calculate the expectation value of energy $\langle E\rangle$ in the state $|\psi(t)\rangle$ i.e. at a later time t.

N.B.You might need

$$p = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right)\ ,\ x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

where $a|n\rangle = \sqrt{n}|n-1\rangle$, $a|0\rangle = 0$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.

3 a) Define a hermitian operator in Quantum Mechanics

b) Show that the eigenvalues of a hermitian operator are real, and give two examples.

c) Define parity in Quantum Mechanics. Is the corresponding operator hermitian?

d) Show that wave functions of opposite parity are orthogonal, and give an example.

e) Show that the wavefunctions of the infinite potential well ($V = 0$ if $|x| < a$, $V = \infty$ otherwise), have definite parity.

4. The wavefunction of a free particle moving in one dimension, at time $t = 0$, is given by

$$\psi(x) = \mathcal{N} \mathrm{e}^{-\frac{1}{2}\alpha\, x^2} \ .$$

where $\mathcal{N}$ and α are positive constants.

a) Evaluate $\mathcal{N}$, in terms of α, by requiring that ψ be normalised.

b) Calculate the dispersions (standard deviations) Δx and Δp, for the state ψ, and verify that they satisfy the Heisenberg principle.

N.B.You might need

$$\int_{-\infty}^{\infty} dx\, x^2\, \mathrm{e}^{-\alpha\, x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\alpha\, x^2} = \sqrt{\frac{\pi}{\alpha}}.$$