Written Exam MFN 1072 Quantum Mechanics

26 June 2013, 2.30–4.30 PM

Please read the following IMPORTANT INSTRUCTIONS

Answer at most TWO questions. A pass is obtained for one complete answer.

You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1.a) Verify the following commutation relations for operators A, B, and C.

$$[AB, C] = A[B, C] + [A, C]B$$

$$[A, BC] = B[A, C] + [A, B]C$$

b) The Hamiltonian for a one-dimensional simple harmonic oscillator of mass m and angular frequency ω is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}$$

c) Assuming the commutation rule

$$[x, p] = i\hbar$$

show that

$$\frac{i}{\hbar}[H, x] = \frac{p}{m}$$

and

$$\frac{i}{\hbar}[H, p] = -m\omega^2 x$$

2. a) In Quantum Mechanics, which operators can be associated to observable physical quantities, and why? Give at least one example.

b) Are the functions

i)

$$\psi(x) = \mathrm{e}^{-\frac{1}{2}\alpha\, x^2} \ .$$

ii)

$$\psi(x) = \mathrm{e}^{i(kx-\omega t)} \ .$$

eigenfunctions of the momentum operator? If so, what are their eigenvalues?

c) What are the possible outcomes of measuring an observable associated to an operator A? How do you express the probabilities of these outcomes?

3 A particle is in an infinitely deep potential well, where $V = 0$ for $|x| \leq a$ and $V = \infty$ for $|x| > a$. Its wave function is $\psi(x) = A \sin kx + B \cos kx$ for $|x| \leq a$, $\psi(x) = 0$ for $|x| > a$, with A, B and k constants.

a) Show that $k = n\pi/(2a)$, $n = 1, 2, 3....$ with $A = 0$ for n odd, $B = 0$ for n even.

b) Show that $\psi(x)$ is an eigenfunction of the Hamiltonian operator, and calculate its energy.

c) Consider the same problem with a shift of origin, so that the well now runs from $x = 0$ to $x = 2a$. What is the form of the wavefunction in this case?

d) The probability density for finding the particle at a given value of x is $|\psi(x)|^2$. Use this fact to *normalise* the wavefunction, i.e. calculate the constants A and B.

e) Calculate the expectation values of position $\langle x \rangle$ and momentum $\langle p \rangle$. *(N.B. By simple reasoning you can deduce the result..)*

Useful formulae

$$\int_{-\infty}^{\infty} dx\, x^2\, \mathrm{e}^{-\alpha\, x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}\ ,$$

$$\int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\alpha\, x^2} = \sqrt{\frac{\pi}{\alpha}}$$