Written Exam CHI 0026 Quantum Mechanics

23 June 2015, 2.00–4.00 PM

Please read the following IMPORTANT INSTRUCTIONS

Answer at most TWO questions. Only two questions will be marked.

You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1. i) In Quantum Mechanics, which operators can be associated to observable physical quantities, and why? Give at least one example.

ii) What are the possible outcomes of measuring an observable associated to an operator? How do you express the probabilities of these outcomes?

iii) Are the functions

a) $\psi_1(x) = \sin kx$

b) $\psi_2(x) = \sin kx - \cos kx$

c) $\psi_3(x) = \cos kx + i \sin kx$

d) $\psi_4(x) = \mathrm{e}^{-\alpha x^2}$

eigenfunctions of the momentum operator? If so, what are their eigenvalues?

2. The wavefunction of a free particle of mass m, moving in one dimension, at time $t = 0$, is given by

$$\psi(x) = \mathcal{N} \mathrm{e}^{-\frac{1}{2}\alpha\, x^2} \ .$$

where $\mathcal{N}$ and α are positive constants.

a) Evaluate $\mathcal{N}$, in terms of α, by requiring that ψ be normalised.

b) Calculate the dispersions (standard deviations) Δx and Δp, for the state ψ, and verify that they satisfy the Heisenberg principle.

N.B.You may need

$$\int_{-\infty}^{\infty} dx\, x^2\, \mathrm{e}^{-\alpha\, x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\alpha\, x^2} = \sqrt{\frac{\pi}{\alpha}}.$$

3.

a) Define a hermitian operator in Quantum Mechanics

b) Show that the eigenvalues of a hermitian operator are real, and give two examples.

c) Define parity in Quantum Mechanics. Is the corresponding operator hermitian?

d) Show that wave functions of opposite parity are orthogonal, and give an example.

e) Show that the wavefunctions of the infinite potential well ($V = 0$ if $|x| < a$, $V = \infty$ otherwise), have definite parity.

4. Assume that, in spherical polar coordinates (r, θ, ϕ), the z-component of angular momentum L_z is represented by $-i\hbar\partial/\partial\phi$, where ϕ is the polar angle, $\phi \in [0, 2\pi]$.

a) Show that L_z has eigenvalues $m\hbar$, with m an integer.

b) The following three states describe the motion of a particle moving in three dimensions

$$\psi_1(r) = \frac{c}{\sqrt{2}}(x + iy) \exp -r^2$$

$$\psi_2(r) = \frac{c}{\sqrt{2}}(x - iy) \exp -r^2$$

$$\psi_3(r) = cz \exp -r^2$$

where c is a normalisation constant and $r^2 = x^2 + y^2 + z^2$.

Verify, using Cartesian coordinates x, y, z, that each of these states is an eigenstate of L_z and find the corresponding eigenvalues.

c) Describe the possible outcomes of measuring L_z for a particle in the state

$$\psi(r) = \frac{c}{\sqrt{5}}(2z - x) \exp -r^2$$

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