Written Exam MFN 1072 Quantum Mechanics

25 September 2012, 2.30–4.30 PM

Please read the following INSTRUCTIONS

A. Answer at most TWO questions. A pass is obtained for one complete answer.
B. You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1 The Hamiltonian for a one-dimensional simple harmonic oscillator of mass m and angular frequency ω is

$$H = \frac{p^2}{2m} + \frac{m\omega^2 x^2}{2}.$$

a) Show that, if $[x,p] = i\hbar$, and $< x(t) >$ and $< p(t) >$ are the expectation values of $x(t)$ and $p(t)$ at time t, they satisfy *precisely* the classical equations, i.e.

$$m\frac{d < x(t) >}{dt} =< p(t) >, \qquad \frac{d < p(t) >}{dt} = -m\omega^2 < x(t) > .$$

b) The state of the oscillator at time $t = 0$ is given by

$$\psi(x) = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} (\frac{\alpha x}{\sqrt{2}} + (\alpha x)^2) \exp{(-\frac{\alpha^2 x^2}{2})}, \quad \alpha = (\frac{m\omega}{\hbar})^{\frac{1}{2}}.$$

and the three lowest energy wavefunctions are

$$\psi_0(x) = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} \exp{(-\frac{\alpha^2 x^2}{2})},$$

$$\psi_1(x) = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} (\sqrt{2}\alpha x) \exp{(-\frac{\alpha^2 x^2}{2})},$$

$$\psi_2(x) = (\frac{\alpha^2}{\pi})^{\frac{1}{4}} \frac{(2\alpha^2 x^2 - 1)}{\sqrt{2}} \exp{(-\frac{\alpha^2 x^2}{2})}.$$

What are their corresponding energies E_0, E_1, E_2?

c) Show that the initial state can be expressed as

$$\psi(x) = A_0\psi_0(x) + A_1\psi_1(x) + A_2\psi_2(x)$$

with A_0, A_1, A_2 constants, and evaluate these constants. What is their meaning?

2 A particle in a potential well $U(x)$ is initially in a state whose wavefunction $\psi(x, 0)$ is an equal-weight superposition of the ground state and first excited state wavefunctions: $\psi(x, 0) = C(\psi_1(x) + \psi_2(x))$.

(CONTINUED)

a) Show that the value $C = \frac{1}{\sqrt{2}}$ normalises $\psi(x, 0)$, assuming that ψ_1 and ψ_2 are themselves normalised.

b) Assuming you know the ground and first excited state energies E_1 and E_2, determine $\psi(x, t)$ at any later time t.

c) Show that the average energy $< E >$ for $\psi(x, t)$ is the arithmetic mean of E_1 and E_2, that is

$$< E > = \frac{E_1 + E_2}{2}.$$

d) Determine the uncertainty ΔE of energy for $\psi(x, t)$.

3 a) Define a hermitian operator in Quantum Mechanics

b) Show that the eigenvalues of a hermitian operator are real, and give two examples.

c) Define parity in Quantum Mechanics. Is the corresponding operator hermitian?

d) Show that wave functions of opposite parity are orthogonal, and give an example.

e) Show that the wavefunctions of the infinite potential well ($V = 0$ if $|x| < a$, $V = \infty$ otherwise), have definite parity.

4 Assume that, in spherical polar coordinates (r, θ, ϕ), the z-component of angular momentum L_z is represented by $-i\hbar\partial/\partial\phi$, where ϕ is the polar angle, in the range $(0, 2\pi)$. Show that L_z has eigenvalues $m\hbar$, with m an integer.

b) The following three states describe the motion of a particle moving in three dimensions

$$\psi_1(r) = \frac{c}{\sqrt{2}}(x + iy) \exp -r^2$$

$$\psi_2(r) = \frac{c}{\sqrt{2}}(x - iy) \exp -r^2$$

$$\psi_3(r) = cz \exp -r^2$$

where c is a normalisation constant and $r^2 = x^2 + y^2 + z^2$.

Verify, using Cartesian coordinates x, y, z, that each of these states is an eigenstate of L_z and find the corresponding eigenvalues.

c) Describe the possible outcomes of measuring L_z for a particle in the state

$$\psi(r) = \frac{c}{\sqrt{5}}(2z - x) \exp -r^2$$