Written Exam MFN 1072 Statistical and Quantum Mechanics

24 September 2013, 2.30–4.30 PM

Please read the following INSTRUCTIONS

Answer at most TWO questions.A pass is obtained for one complete answer
You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

1. A particle in a potential well $U(x)$ is initially in a state whose wavefunction $\psi(x,0)$ is an equal-weight superposition of the ground state and first excited state wavefunctions: $\psi(x,0) = C(\psi_1(x) + \psi_2(x))$.
a) Show that the value $C = \frac{1}{\sqrt{2}}$ normalises $\psi(x,0)$, assuming that ψ_1 and ψ_2 are themselves normalised.

b) Assuming you know the ground and first excited state energies E_1 and E_2, determine $\psi(x,t)$ at any later time t.

c) Show that the average energy $< E >$ for $\psi(x,t)$ is the arithmetic mean of E_1 and E_2, that is

$$< E >= \frac{E_1 + E_2}{2}.$$

d) Determine the uncertainty ΔE of energy for $\psi(x,t)$.

2. Assume that, in spherical polar coordinates (r, θ, ϕ), the z-component of angular momentum L_z is represented by $-i\hbar\partial/\partial\phi$, where ϕ is the polar angle, in the range $(0, 2\pi)$.
a) Show that L_z has eigenvalues $m\hbar$, with m an integer.

b) The following three states describe the motion of a particle moving in three dimensions

$$\psi_1(r) = \frac{c}{\sqrt{2}}(x + iy) \exp -r^2$$

$$\psi_2(r) = \frac{c}{\sqrt{2}}(x - iy) \exp -r^2$$

$$\psi_3(r) = cz \exp -r^2$$

where c is a normalisation constant and $r^2 = x^2 + y^2 + z^2$.

Verify, using Cartesian coordinates x, y, z, that each of these states is an eigenstate of L_z and find the corresponding eigenvalues.

c) Describe the possible outcomes of measuring L_z for a particle in the state

$$\psi(r) = \frac{c}{\sqrt{5}}(2z - x) \exp -r^2$$

3. The wavefunction of a free particle of mass m, moving in one dimension, at time $t = 0$, is given by

$$\psi(x) = \mathcal{N} \mathrm{e}^{-\frac{1}{2}\alpha\, x^2} \ .$$

where $\mathcal{N}$ and α are positive constants.

a) Evaluate $\mathcal{N}$, in terms of α, by requiring that ψ be normalised.

b) Calculate the dispersions (standard deviations) Δx and Δp, for the state ψ, and verify that they satisfy the Heisenberg principle.

N.B. You may need

$$\int_{-\infty}^{\infty} dx\, x^2\, \mathrm{e}^{-\alpha\, x^2} = \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}}, \quad \int_{-\infty}^{\infty} dx\, \mathrm{e}^{-\alpha\, x^2} = \sqrt{\frac{\pi}{\alpha}}.$$

4. Prove the following results for an eigenstate of the Hamiltonian of a one-dimensional simple harmonic oscillator

a) The expectation values of the position and momentum are zero

b) The expectation values of the potential and kinetic energies are equal

c) The uncertainties in position and momentum Δx and Δp, satisfy the relation $\Delta x \Delta p = (n + \frac{1}{2})\hbar$, where n is the quantum number of the state.

d) The ground state eigenfunction is a Gaussian, and the parity of the state n is even for n even, and odd for n odd.

N.B. You may need

$$p = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right) \ , \ x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

where $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.