Written Exam MFN 1072 Quantum Mechanics

16 September 2014, 2.00–4.00 PM

Please read the following IMPORTANT INSTRUCTIONS

Answer at most TWO questions. Only two questions from each section will be marked. A pass is obtained for one complete answer.

You may not use notes or textbooks, but the lecture notes etc are available for consultation at the front desk.

Answer at most TWO questions

1.

a) If O is a quantum–mechanical operator, what is the definition of the corresponding Hermitian conjugate operator $O^\dagger$?

b) Define what is meant by a Hermitian operator in quantum mechanics

c) Show that d/dx is not a Hermitian operator. What is its Hermitian conjugate, $(d/dx)^\dagger$?

d) Prove that, for any two operators A and B, $(AB)^\dagger = B^\dagger A^\dagger$.

2. A particle in a potential well $U(x)$ is initially in a state whose wavefunction $\psi(x,0)$ is an equal-weight superposition of the ground state and first excited state wavefunctions: $\psi(x,0) = C(\psi_1(x) + \psi_2(x))$.

a) Show that the value $C = \frac{1}{\sqrt{2}}$ normalises $\psi(x,0)$, assuming that ψ_1 and ψ_2 are themselves normalised.

b) Assuming you know the ground and first excited state energies E_1 and E_2, determine $\psi(x,t)$ at any later time t.

c) Show that the average energy $< E >$ for $\psi(x,t)$ is the arithmetic mean of E_1 and E_2, that is

$$< E >= \frac{E_1 + E_2}{2}.$$

d) Determine the uncertainty ΔE of energy for $\psi(x,t)$.

(CONTINUED)

3. a) Define the single particle angular momentum operators L_x, L_y, L_z in terms of the position and momentum operators for the particle.

b) Assuming $[x, p_x] = i\hbar$ etc, show that

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_z, L_x] = i\hbar L_y$$

c) Defining $L^2 = {L_x}^2 + {L_y}^2 + {L_z}^2$, show that

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0,$$

d) Show that the eigenvalues of L^2 are all positive or zero.

e) A point particle moving in three dimensions is described by a wave function

$$\phi(\vec{r}) = cz \exp\left(-\alpha r^2\right)$$

where c and α are constants and $r^2 = x^2 + y^2 + z^2$. Assuming that $\phi(\vec{r})$ is an eigenfunction of L^2, is it also an eigenfunction of L_z? If so, what is its eigenvalue?

4. Prove the following results for an eigenstate of the Hamiltonian of a one-dimensional simple harmonic oscillator

a) The expectation values of the position and momentum are zero

b) The expectation values of the potential and kinetic energies are equal

c) The uncertainties in position and momentum Δx and Δp, satisfy the relation $\Delta x \Delta p = (n + \frac{1}{2})\hbar$, where n is the quantum number of the state.

d) The ground state eigenfunction is a Gaussian, and the parity of the state n is even for n even, and odd for n odd.

(You may need

$$p = -\mathrm{i}\sqrt{\frac{m\omega\hbar}{2}}\left(a - a^\dagger\right) \ , \ x = \sqrt{\frac{\hbar}{2m\omega}}\left(a + a^\dagger\right)$$

where $a|n\rangle = \sqrt{n}|n-1\rangle$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$.*)*