Wave statistics in unimodal and bimodal seas from a second-order model

Alessandro Toffoli a,*, Miguel Onorato b, Jaak Monbaliu a

a Hydraulics Laboratory, Katholieke Universiteit Leuven, Kasteelpark Arenberg 40, Heverlee 3001, Belgium
b Dip. Fisica Generale, Università di Torino, Via P. Giuria 1, 10125 Torino, Italy

Received 1 November 2005; received in revised form 11 January 2006; accepted 17 January 2006
Available online 5 June 2006

Abstract

A second-order surface wave model is used to investigate the effects of the spectral distribution on the statistical properties of the surface elevation. To this end single and double peaked directional wave spectra are considered at different water depths. For unimodal seas (i.e. single peaked), the addition of directional components reduces the effects of the second-order interactions in deep water and increases them in shallower depths. For a bimodal sea (i.e. double peaked), on the other hand, a large angle between the wave trains decreases systematically the vertical asymmetry of the wave profile. However, the nonlinear interactions seem to reach their maximum strength when the two wave spectra are slightly separated in direction. This produces an evident deviation of the wave crest distribution at low probability levels if compared with the unimodal condition.

© 2006 Elsevier SAS. All rights reserved.

Keywords: Wave statistics; Directional distribution; Nonlinear waves; Finite water depth; Second-order surface wave model; Bimodal spectrum

1. Introduction

For a long time the Gaussian theory has been used to describe the statistics of surface gravity waves. This theory, however, assumes that the waves are a linear phenomenon [1]. Hence it fails to account for nonlinear wave effects such as the sharpening of the crests and the flattening of the troughs. Considering that waves are weakly nonlinear, several approaches have already been proposed [2–4] to describe the statistical properties of ocean wave trains. Unfortunately, many of them have been developed for the narrowband unidirectional case, and might not be generalized for a broadband multidirectional wave field. To achieve a more general description, a theory for the second-order interactions of waves in a random directional sea has been derived by Sharma and Dean [5] for arbitrary water depths, as an extension of the theory developed by Longuet-Higgins [2] for narrowband unidirectional waves. In principle, this approach is able to capture the effects of wave steepness, water depth, and directional spreading with no approximation other than the truncation of the small amplitude expansion to the second order.

At the second-order in a Stokes expansion, the crests are sharper and higher, while the troughs are flatter and lower; there is also a long wave set-down (see e.g. [6]), which is expected to depress the mean free surface in regions

* Corresponding author. Tel.: +32 16 321174; fax: +32 16 321989.
E-mail address: alessandro.toffoli@bwk.kuleuven.be (A. Toffoli).
of groups of high waves. As discussed by Dalzell [7], the magnitude of the set-down for a given significant wave height, peak period, and water depth could reduce substantially if the waves are short-crested, i.e. multidirectional. In case the nonlinearity is sufficiently high, this might also give rise to higher crests. Supported by in-situ measurements, Forristall [8] looked at the distribution of simulated second-order long-crested (i.e. unidirectional) and short-crested waves. One of the main findings confirms to some extent that a directional wave field produces higher crests than a unidirectional system as the water depth decreases (see also [9]).

These previous studies considered the existence of only one spectral peak. Generally speaking, however, two different types of waves usually characterize the sea surface: wind sea and swell. A recent work based on the analysis of some 270 ship accidents [10] has revealed that the existence of wave trains traveling along different directions, also known as mixed, combined or crossing seas, may preclude the safety of maritime activities.

There are many mechanisms that can lead to the formation of extreme waves and to a different shape of the probability density function (e.g. nonlinear interactions and modulational instability [11–14], wave–current interactions [15,16], linear Fourier superposition, see [17] for a review); in the present study we investigate the effect of the second-order nonlinearity. Our choice is not due to the fact that we believe that second order mechanism is the most relevant mechanism for the formation of extreme waves, but just because in our opinion, in order to have a global comprehension of extreme events, each mechanism should be studied in details.

Therefore, in the present paper, double-peak directional wave spectra are investigated using a standard second-order model [5] in arbitrary depth. Many random simulations of seventeen-minutes time series have been performed, since convergence of the statistics for rare events is achieved by a large number of repetitions. This would, in principle, allow one to verify whether there exists an angle between the two spectral peaks that might be associated to a higher probability for the formation of extreme waves. Since a correct knowledge of the wave crest statistics (i.e. highest point of a wave trace between a zero up-crossing and the consecutive zero down-crossing) is often needed for design purposes, we will focus our analysis on the probability distribution of crest elevation.

In the first section of this paper we introduce the second-order wave model and its numerical implementation. We also discuss in details the different directional distributions characterizing the input wave spectra. In Section 3 we describe the behavior of the skewness obtained from second-order simulations for different sea state conditions; we will show that there are some conditions characterized by double-peak spectra for which the wave skewness reaches larger values compared to the unidirectional case. We illustrate in Section 4 the behavior of the cumulative probability density function for wave crests for low probability levels. Conclusions are reported in Section 5.

2. Second-order wave profiles

2.1. The wave model

Water waves are normally described in terms of a velocity potential \( \Phi(x, y, z, t) \). In case of uniform water depth \( h \), the potential and the surface elevation \( \eta(x, y, t) \) are determined by the following boundary value problem (see for example [18]):

\[
\nabla^2 \Phi = 0, \tag{1}
\]

\[
\frac{\partial \Phi}{\partial t} + \frac{1}{2}(\nabla \Phi)^2 + gz = 0 \quad \text{at} \quad z = \eta(x, y, t), \tag{2}
\]

\[
\frac{\partial \eta}{\partial t} + \frac{\partial \Phi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial \Phi}{\partial y} \frac{\partial \eta}{\partial y} - \frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = \eta(x, y, t), \tag{3}
\]

\[
\frac{\partial \Phi}{\partial z} = 0 \quad \text{at} \quad z = -h. \tag{4}
\]

Eq. (1) is the Laplace equation and Eqs. (2), (3) and (4) are respectively the dynamic free surface boundary condition, the kinematic free surface boundary condition, and the bottom boundary condition. A solutions of the system (1)–(4) can be sought using the following expansion (see e.g. [18]):

\[
\begin{align*}
\Phi &= \Phi^{(1)} + \Phi^{(2)} + \cdots \\
\eta &= \eta^{(1)} + \eta^{(2)} + \cdots 
\end{align*}
\]

where \( \frac{\Phi^{(n+1)}}{\Phi^{(n)}} = \frac{\eta^{(n+1)}}{\eta^{(n)}} = \mathcal{O}(\epsilon). \]
Here \(\epsilon\) is a small parameter in the expansion (5) and it is typically proportional to the wave steepness \(\xi = ka\), where \(k = 2\pi/L\) (\(L\) = wavelength) is the wave number, and \(a\) is the wave amplitude equal to half the wave height. Although the steepness already provides a measure of the wave nonlinearity, it may be inadequate at arbitrary depths. In such a case, it is more appropriate to define the following generalized nonlinear parameter (see [19] for details) which is valid for any depth:

\[
\mu = ka \cdot \coth(kh) \cdot \left( 1 + \frac{3}{2 \sinh^2(kh)} \right).
\]

(6)

As it involves both the steepness (\(\xi\)) and the relative depth (\(kh\)), the parameter \(\mu\) provides an unique description of the nonlinearity. In deep water, \(kh \to \infty\), \(\mu\) tends to the steepness and, as \(kh \to 0\), \(\mu\) tends the Ursell number [20].

For a sea state characterized by a certain spectral density function \(E(\omega, \vartheta)\), it is straightforward to show that a first order solution of the system (1)–(4) takes the following form:

\[
\eta^{(1)}(x, t) = \sum_{i=1}^{N} \sum_{l=1}^{M} a_{il} \cos[k_i(x \cos \vartheta_l + y \sin \vartheta_l) - \omega_i t + \epsilon_{il}],
\]

(7)

where \(t\) is the time, \(x = (x, y)\) is position vector, \(\omega_i\) is the angular frequency, \(\vartheta_l\) is the direction and \(\epsilon_{il}\) is the phase; \(k_i\) is related to frequency through the linear dispersion relation \(\omega_i = \sqrt{gk_i \tanh(k_i h)}\); \(N\) is the total number of frequencies and \(M\) is the total number of directions considered in the model; \(a_{il}\) are the spectral amplitudes, which are calculated as follow:

\[
a_{il} = a(\omega_i, \vartheta_l) = \sqrt{2E(\omega_i, \vartheta_l) \Delta \omega \Delta \vartheta}.
\]

(8)

The second-order correction to the linear wave surface (7) as calculated by Sharma and Dean [5] is

\[
\eta^{(2)}(x, t) = \frac{1}{4} \sum_{i,j=1}^{M} \sum_{l,m=1}^{M} a_{il} a_{jm} [K_{ijlm}^+ \cos(\varphi_{il} - \varphi_{jm}) + K_{ijlm}^- \cos(\varphi_{il} + \varphi_{jm})],
\]

(9)

where \(\varphi_{il} = k_i(x \cos \vartheta_l + y \sin \vartheta_l) - \omega_i t + \epsilon_{il}\), and \(K_{ijlm}^+\) and \(K_{ijlm}^-\) are the interaction kernels. The positive kernel is responsible for the sharpness of the crests and flatness of the troughs, the negative kernel expresses the set-down beneath energetic parts and the set-up between these regions. The full expression for the interaction kernels can be found in Eqs. (34)–(37) of [5], and also in Eqs. (12)–(19) of [8]. Here, for completeness, we provide the expression for the kernels in Appendix A.

### 2.2. The input spectra

In real sea conditions the wave energy not only is distributed in the frequency domain but also in direction. The wave spectrum is therefore defined as \(E(\omega, \vartheta) = S(\omega)D(\omega, \vartheta)\), where \(S(\omega)\) represents the distribution in the frequency domain, and \(D(\omega, \vartheta)\) is the directional spreading function.

When the sea state is characterized by one single peak (unimodal spectrum), the directional spreading of the system is usually defined by the directional spread coefficient \(\sigma_1\), which corresponds to the standard deviation of the directional function [22]. Beside this, however, a certain sea state may contain two independent wave trains (e.g. a wind sea and a swell or two swells) with different directional distributions and mean directions (i.e. bimodal spectrum). In this case, also the angle \(\delta\) between the two systems must be accounted for in order to fully describe a directional sea state.

We first consider unimodal sea conditions to study the effects of the directional spreading. To generate a two-dimensional spectrum, the energy in the frequency domain is distributed according to a JONSWAP spectrum (see e.g. [23]) with peak period \(T_p = 12\) s, peak enhancement factor \(\gamma = 3.3\), and Phillips parameter \(\alpha = 0.019\), which corresponds to a significant wave height \(H_{m0} = 11\) m. For completeness, the expression for the JONSWAP spectrum used was:

\[
S(\omega) = \frac{\alpha}{(\omega/2\pi)^5} \exp \left[ - \frac{5}{4} \left( \frac{\omega_p}{\omega} \right)^4 \right] \gamma \exp \left[ \frac{(\omega/2\pi - \omega_p/2\pi)^2}{2\sigma_0^2(\omega_p/2\pi)^2} \right].
\]

(10)
where \( \omega_p = 2\pi / T_p \) is the peak angular frequency. The parameter \( \sigma_0 \) is equal to 0.07 when \( \omega \leq \omega_p \) and to 0.09 when \( \omega \geq \omega_p \).

For the distribution of the energy in the directional domain, we consider the \( \cos-2s \) function (see e.g. [21,22] for more details). This distribution has the following form:

\[
D(\omega, \theta) = \frac{\Gamma(s(\omega) + 1)}{2\sqrt{\pi} \Gamma(s(\omega) + 1/2)} \cos^{2s(\omega)} \left( \frac{\theta}{2} \right),
\]

(11)

where \( \Gamma \) denotes the Gamma function [24], \( \theta \) is the azimuth measured counterclockwise from the principal wave direction, and \( s(\omega) \) is a function of frequency describing the magnitude of the directional spreading. Goda [21] defines \( s(\omega) \) as follow:

\[
\begin{align*}
  s(\omega) = \left( \frac{\omega}{\omega_p} \right)^5 s_{max} & \quad \text{if } \omega \leq \omega_p, \\
  s(\omega) = \left( \frac{\omega}{\omega_p} \right)^{-2.5} s_{max} & \quad \text{if } \omega > \omega_p.
\end{align*}
\]

(12)

If \( s_{max} \to \infty \) a narrow directional distribution characterizes the spectrum and waves are long-crested, while if \( s_{max} \to 0 \) a broad distribution is generated and waves are short-crested. For the present study, we assume the following four values: (i) \( s_{max} = 100 \) corresponding to an almost unidirectional system; (ii) \( s_{max} = 75 \) corresponding to swells with long decay distance; (iii) \( s_{max} = 25 \) corresponding to swells with short decay distance; and (iv) \( s_{max} = 10 \) corresponding to wind waves. The concurrent directional spread coefficients can be calculated directly from \( s_{max} \), using the following expression (see [22]):

\[
\sigma_1 = \sqrt{2 / s_{max} + 1}.
\]

(13)

Table 1 reports the values of \( s_{max} \) and the related directional spread coefficients used herein. A graphical description of the different wave spectra is also given in Fig. 1.

We then consider the case of a bimodal spectrum to study the effects on the statistical properties of the surface elevation of wave trains traveling in different directions. For consistency with the previous case, we use a sea state with equivalent significant wave height (\( H_m 0 = 11 \) m) and peak period (\( T_p = 12 \) s). In contrast, however, we distribute the energy over two independent but identical directional spectra \( E_1(\omega, \theta - \bar{\theta}_1) \) and \( E_2(\omega, \theta - \bar{\theta}_2) \), where \( \bar{\theta}_1 \) and \( \bar{\theta}_2 \) are the mean directions of propagation of the two different wave systems. In the frequency domain, the energy density is defined using a JONSWAP formulation with \( T_p = 12 \) s, \( \gamma = 3.3 \), and \( \alpha = 0.0095 \). The Phillips parameter \( \alpha \) is set equal to the half one used for the unimodal spectra, in order to satisfy the following condition:

\[
\int_0^\infty \int_0^\infty E_1(\omega, \theta - \bar{\theta}_1) \, d\omega \, d\theta = \int_0^\infty \int_0^\infty E_2(\omega, \theta - \bar{\theta}_2) \, d\omega \, d\theta = \frac{1}{2} \int_0^\infty \int_0^\infty E(\omega, \theta) \, d\omega \, d\theta.
\]

(14)

To minimize the influence of the directional spreading function, both spectra are assumed to be narrow, and hence a \( \cos-2s \) function with \( s_{max} = 100 \) is accounted for. The spectra are then centered around the two different mean wave directions. We have performed different simulations with different angles \( \delta = \bar{\theta}_1 - \bar{\theta}_2 \); we have used the following values: (i) \( \delta = 35^\circ \) (0.61 rad); (ii) \( \delta = 55^\circ \) (0.96 rad); (iii) \( \delta = 75^\circ \) (1.31 rad); and (iv) \( \delta = 90^\circ \) (1.57 rad). If \( \delta \to 0 \)

<table>
<thead>
<tr>
<th>( s_{max} )</th>
<th>( \sigma_1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.140</td>
</tr>
<tr>
<td>75</td>
<td>0.162</td>
</tr>
<tr>
<td>25</td>
<td>0.277</td>
</tr>
<tr>
<td>10</td>
<td>0.426</td>
</tr>
</tbody>
</table>
the spectrum assumes a unimodal form with a directional spreading $s_{\text{max}} = 100$ ($\sigma_1 = 0.140$). The different bimodal spectra are illustrated in Fig. 2.

The effects of the directional spreading may vary with the water depth. Therefore, we have considered wave spectra at several water depths. We use a constant $H_s$ and $T_p$ for all simulations; this automatically produces a value of the nonlinear parameter $\mu$ (see Eq. (6) where the wave number $k$ is related to the peak period and the amplitude $a$ to the significant wave height). The different depths, relative depths, and nonlinearities that are taken into account are reported in Table 2.

### 2.3. The simulations

In order to produce stable statistics at low probability levels (e.g. $10^{-4}$), we perform many simulations of $\eta(x, t)$ by using the same spectral shape. To this end, a first-order contribution $\eta^{(1)}(x, t)$ is initially calculated from Eq. (7) by choosing the phase $\varepsilon$ from a uniform random distribution in the interval $[0, 2\pi]$, and using an inverse Fast Fourier Transform to perform the summations in (7). Subsequently, the second order contribution (i.e. $\eta^{(2)}(x, t)$) is calculated for each pair of wave components using the summations in (9). The addition of these two contributions then gives a second-order time series. For each wave spectrum (Figs. 1 and 2) and water depth (Table 2), we compute a total of 1250 repetitions with 2048 time steps at sampling frequency of 2 Hz and with angular resolution of 12° (i.e. 30 directions). Note that the simulations are performed at a fixed location (for convenience we have selected $x = [0, 0]$). Considering that the mean period for a JONSWAP spectrum with $\gamma = 3.3$ is $0.8T_p$ [21], these repetitions will lead to

---

**Table 2**

<table>
<thead>
<tr>
<th>Depth $d$ (m)</th>
<th>Relative $kd$</th>
<th>Nonlinearity $\xi$</th>
<th>Nonlinearity $\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>1.737</td>
<td>0.159</td>
<td>0.203</td>
</tr>
<tr>
<td>50</td>
<td>1.486</td>
<td>0.163</td>
<td>0.243</td>
</tr>
<tr>
<td>40</td>
<td>1.245</td>
<td>0.171</td>
<td>0.322</td>
</tr>
<tr>
<td>30</td>
<td>1.013</td>
<td>0.186</td>
<td>0.497</td>
</tr>
</tbody>
</table>

---

**Fig. 1. Schematic representation of unimodal spectra:** (a) $s_{\text{max}} = 100$; (b) $s_{\text{max}} = 75$; (c) $s_{\text{max}} = 25$; (d) $s_{\text{max}} = 10$. 

---

The spectrum assumes a unimodal form with a directional spreading $s_{\text{max}} = 100$ ($\sigma_1 = 0.140$). The different bimodal spectra are illustrated in Fig. 2.

The effects of the directional spreading may vary with the water depth. Therefore, we have considered wave spectra at several water depths. We use a constant $H_s$ and $T_p$ for all simulations; this automatically produces a value of the nonlinear parameter $\mu$ (see Eq. (6) where the wave number $k$ is related to the peak period and the amplitude $a$ to the significant wave height). The different depths, relative depths, and nonlinearities that are taken into account are reported in Table 2.

### 2.3. The simulations

In order to produce stable statistics at low probability levels (e.g. $10^{-4}$), we perform many simulations of $\eta(x, t)$ by using the same spectral shape. To this end, a first-order contribution $\eta^{(1)}(x, t)$ is initially calculated from Eq. (7) by choosing the phase $\varepsilon$ from a uniform random distribution in the interval $[0, 2\pi]$, and using an inverse Fast Fourier Transform to perform the summations in (7). Subsequently, the second order contribution (i.e. $\eta^{(2)}(x, t)$) is calculated for each pair of wave components using the summations in (9). The addition of these two contributions then gives a second-order time series. For each wave spectrum (Figs. 1 and 2) and water depth (Table 2), we compute a total of 1250 repetitions with 2048 time steps at sampling frequency of 2 Hz and with angular resolution of 12° (i.e. 30 directions). Note that the simulations are performed at a fixed location (for convenience we have selected $x = [0, 0]$). Considering that the mean period for a JONSWAP spectrum with $\gamma = 3.3$ is $0.8T_p$ [21], these repetitions will lead to
approximately 130 000 waves for each selected sea state. An additional set of 1250 repetitions was made considering an unidirectional spectrum (JONSWAP) with $T_p = 12$ s, $\gamma = 3.3$, and $\alpha = 0.019$ to look at the differences induced by the directional distributions.

Note that repeating a run with precisely the same spectral variance may not include all the natural variability of waves since the variances are actually random variables with a Chi-squared distribution [8]. However if a directional sea is simulated, the addition of different directional components, each with a random phase, at the same frequency automatically restores this variability [8]. Therefore, a random variable for the spectral variances is not accounted for in the three-dimensional simulations.

The interaction between long and very short waves is not properly calculated with a truncated Stokes expansion [25]. As this may generate waves with an unphysical shape [9], a small part of the low-frequency and high-frequency spectral energy can be removed to avoid a relatively large error of convergence of the nonlinear interaction. Prevosto [9] performed many tests by truncating several fractions between 1% and 10% of the energy at the low and high tails of the spectrum. The results showed that the 1% truncation produces a measurable effect on crests heights for the unidirectional simulations, but has only a marginal influence for the directionally spread simulations. Herein we consider a 1.5% truncation for the three-dimensional experiments as this would also reduce significantly the computational time.

3. Vertical asymmetry

The sharpening of the wave crests and the flattening of the wave troughs are the most obvious manifestations of nonlinearity. In a time series, these effects appear as the vertical asymmetry of the signal, which can be measured using the skewness [1].

The vertical asymmetry increases as the nonlinear parameter $\mu$ increases. However, modifications of the skewness can also be expected because of the directional spreadings. In order to describe this influence, we define a parameter $\lambda'$ as the ratio of the skewness of a directional sea, $\lambda_{3\text{ dir}}$, to the skewness of a unidirectional sea, $\lambda_{3\text{ uni}}$:

$$
\lambda' = \frac{\lambda_{3\text{ dir}}}{\lambda_{3\text{ uni}}}.
$$

(15)
In Fig. 3 the behavior of the parameter $\lambda'$ as a function of the directional spread coefficient for several levels of nonlinearity is shown for unimodal seas. We also provide a linear fit to the data. For small values of our nonlinear parameter $\mu$ the addition of the directional components reduces the skewness. The result, caused by the slightly lower values of the second-order interaction kernel for waves that are not colinear, is consistent with the theoretical work conducted by Longuet-Higgins [2] and the numerical work performed by Prevosto [9]. For constant $\sigma_1$, as the nonlinear parameter is increased, we observe in general an increase of $\lambda'$. Moreover, we observe that for $\mu = 0.154$, $\mu = 0.203$ and $\mu = 0.243$, $\lambda'$ is always lower than one and decreases as the spread coefficient $\sigma_1$ increases. For $\mu = 0.322$ ($h = 40$ m, Table 2), the vertical asymmetry remains approximately constant for increasing directionality. For the highest level of nonlinearity considered ($\mu = 0.497$), which corresponds to shallower depths in our simulations, this trend is inverted and larger values of $\lambda'$ are observed as $\sigma_1$ increases. It should be here mentioned that this result should be considered with some caution because for large nonlinearity, as the one reached in shallow water, second-order models could not be sufficient to reproduce the physics and higher order models should be used.

We now consider a bimodal sea. In Fig. 4 we show $\lambda'$ as a function of $\delta$ for different values of $\mu$. Similarly to the unimodal case (but for low nonlinear levels) a wide angle, i.e. $\delta \to 1.57$ rad ($90^\circ$), produces a reduction of the vertical asymmetry (see also [2]). However, when a bimodal spectrum is characterized by a narrow aperture, i.e. $\delta = 0.61$ rad ($35^\circ$), the wave profile assumes a more skewed shape than any other directional sea with equivalent level of nonlinearity. It is also more skewed than for an unimodal sea. Moreover, for high values of the nonlinear coefficient $\mu$, a narrow aperture bimodal spectrum gives a skewness up to 25\% larger than for the unidirectional case.

In a second-order expansion, the interaction kernels $K^+$ and $K^-$ provide a positive and a negative contribution to the skewness respectively [8]. Whereas the first makes the crests sharper and the troughs flatter, the second depresses the mean sea level (set-down) under the energetic groups, limiting the growth of vertical asymmetry. The surface elevation, consequently, can be rewritten as follows: $\eta = \eta_1^{(1)} + \eta_1^{(2)} + \eta_1^{(2)}$, where $\eta_1^{(2)}$ is due to the positive kernel.
and \( \eta_{(2)} \) is due to the negative kernel (see Eq. (9)). In order to illustrate the negative contribution that the set-down produces on the total skewness, a second parameter \( \lambda'' \) is calculated as the ratio of the skewness of \( \bar{\eta} = \eta^{(1)} + \eta_{(2)} \) to the skewness of \( \eta \). When \( \lambda'' \to 1 \) no contribution is given, while \( \lambda'' > 1 \) if a negative contribution is provided. The qualitative trends describing \( \lambda'' \) as a function of the directional spreading, and the nonlinearity are presented in Figs. 5 and 6. In agreement with theoretical studies [7], the magnitude of the set-down decreases for large directional spreadings. In Fig. 7 we show the slopes of the curves in Figs. 5 and 6. This slope remains relatively constant with growing nonlinearity if bimodal seas are simulated and it enhances rapidly when unimodal conditions are considered. In other words, the attenuation of the set-down due to the directional distribution in an unimodal sea is much stronger in finite water depth than in deep water.

In general, the set-down tends to be small or even nil for low nonlinearity and large directional spreadings (see Fig. 5, and [7]). However, it is interesting to note, in that respect, that for broad aperture bimodal seas the negative kernel can give rise to a set-up in energetic regions instead of the expected set-down, see \( \eta_{(2)} \) in the lower panel of Fig. 8.

4. Wave crest distribution

4.1. Unimodal seas

A second-order expansion of the sea surface makes the wave crests sharper and higher than expected by linear theory [18]; in Figs. 9 and 10 we show the cumulative distribution for wave crests for \( \mu = 0.154 \) and \( \mu = 0.497 \). In deep water (\( \mu = 0.154 \)) the addition of the different directional components reduces the strength of the nonlinear interactions (see e.g. Fig. 3), and hence limits the rising of the crest elevation. At a low probability level (e.g. \( 10^{-4} \))
Fig. 7. Gradient of the set-down reduction ($\nu$) due to directional spreading as a function of the nonlinear coefficient $\mu$.

Fig. 8. Wave elevation ($\eta = \eta^{(1)} + \eta^{(2)} + \eta^{(2)}$) and low-frequency second-order response ($\eta^{(2)}_r$) for a bimodal sea with $\delta = 90^\circ$ in deep water.

| Table 3 | $H_{Cr}$ unimodal/$H_{Cr}$ unidirectional for different unimodal sea conditions, and nonlinear levels: average values for the probability level of $10^{-4}$ |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                | $\mu = 0.154$  | $\mu = 0.203$  | $\mu = 0.243$  | $\mu = 0.322$  | $\mu = 0.497$  |
| $\sigma_1 = 0.140$ | 0.968          | 0.968          | 0.986          | 1.010          | 1.021          |
| $\sigma_1 = 0.162$ | 0.963          | 0.997          | 1.004          | 1.013          | 1.024          |
| $\sigma_1 = 0.277$ | 0.968          | 0.978          | 1.003          | 1.010          | 1.034          |
| $\sigma_1 = 0.426$ | 0.955          | 0.964          | 0.995          | 1.000          | 1.026          |

the three-dimensional simulations (multidirectional) yield crest heights that are about 4% lower than the unidirectional condition (see also [8]). As the water depth becomes shallower ($\mu = 0.497$), the limiting effects induced by the directional spreading reduces. If the nonlinearity of the system, then, is sufficiently high, the crest elevation rises higher than in a unidirectional sea surface. For a relative depth corresponding to a nonlinear level $\mu = 0.497$, the multidirectional simulations produces crests that are on average 2.6% higher than unidirectional simulations. This can be read in Table 3, where the ratio $H_{Cr}$ unimodal/$H_{Cr}$ unidirectional, with $H_{Cr}$ = crests height, is reported for the different $\sigma_1$ and $\mu$. Despite the fact that the presence of several directional components affects the probability density function of the crest elevation, the form of the distribution seems to be rather insensitive to the variation of the directional spread coefficient $\sigma_1$.

4.2. Bimodal seas

A recent study [10] has shown that bimodal seas my preclude the safety of maritime activities. Apart from some difficulties related to ship manoeuvring, it should not be excluded that mixed seas might also contribute to generate wave crests, which are larger than the ones expected in an unimodal sea.

The investigation of the vertical asymmetry from the second-order directional simulations indicates, however, a significant reduction of the strength of the nonlinear interactions for large aperture of the angle $\delta$. In Figs. 11 and 12 we show the cumulative probability density function for wave crests; for $\delta = 1.57$ rad ($90^\circ$) and probability level of $10^{-4}$ the crests elevations are about 7% lower than a unidirectional case for low nonlinear levels, and about 3% lower for high nonlinear levels. Comparing with the unimodal case (i.e. $\delta \to 0$), the crests show a slightly limited deviation, which is on average 3% (values between brackets in Table 4). If the aperture $\delta$ reduces (i.e. $\delta = 1.31$ rad ($75^\circ$) or $\delta = 0.97$ rad ($55^\circ$)), the vertical asymmetry enhances, though the crests do not rise higher than the unimodal simulations (Table 4).

When the wave trains assume relatively close mean directions (i.e. $\delta = 0.61$ rad ($35^\circ$)), the second-order interactions grow much stronger than in an unimodal sea condition (see e.g. Fig. 4). This consequently leads to a visible departure of the wave crest distribution (triangle dots in Figs. 11 and 12). On average, the crests tend to be 4.5% higher than in a single-peaked sea state at the probability level of $10^{-4}$. This difference, however, reaches a maximum for low ($\mu = 0.154$) and high ($\mu = 0.497$) nonlinearity, while it slightly decreases within these levels.
In order to measure whether a bimodal sea state with $\delta = 35^\circ$ produces a wave crest distribution that is significantly different from the one related to an unimodal sea ($\delta \rightarrow 0$), we estimate the confidence limits of the latter by means of a bootstrap technique. This method is a resampling procedure, which yields random copies of the original data set (see [26] and references therein). For each bootstrap sample, the statistical properties, i.e. wave crest distribution, can be recomputed; repeating this process many times (typically 1000), the asymptotic 95% confidence interval for the probability density function can be evaluated consequently. The wave crest distribution of the unimodal sea state and the concurrent confidence limits are presented in Fig. 13 for $\mu = 0.154$ and $\mu = 0.497$; the distribution of the bimodal

In order to measure whether a bimodal sea state with $\delta = 35^\circ$ produces a wave crest distribution that is significantly different from the one related to an unimodal sea ($\delta \rightarrow 0$), we estimate the confidence limits of the latter by means of a bootstrap technique. This method is a resampling procedure, which yields random copies of the original data set (see [26] and references therein). For each bootstrap sample, the statistical properties, i.e. wave crest distribution, can be recomputed; repeating this process many times (typically 1000), the asymptotic 95% confidence interval for the probability density function can be evaluated consequently. The wave crest distribution of the unimodal sea state and the concurrent confidence limits are presented in Fig. 13 for $\mu = 0.154$ and $\mu = 0.497$; the distribution of the bimodal
Fig. 13. Bootstrap uncertainty estimate for the wave crest distribution of an unimodal sea state.

sea ($\delta = 35^\circ$) is also included. As the latter lays outside the upper bound of the confidence interval, the bootstrap uncertainty confirms to some extent that conditions of mixed seas with relative narrow angle between the spectral peaks generate wave crest statistics which deviates significantly from a single peaked sea state.

5. Conclusions

This study has investigated the effect of the second-order interaction in directionally spread sea states on the statistical properties of surface gravity waves. To this end, unimodal and bimodal directional spectra have been used as input to simulated wave time series with a second-order model. A typical simulation has 2048 time steps with a sampling frequency of 2 Hz. In order to achieve a reliable statistics at low probability levels, 1250 repetitions using different random phases were performed.

The addition of different directional components reduces, in general, the strength of the second-order interaction. Considering the tail of statistical distribution (e.g. $10^{-4}$), these effects generate crest elevations which are about 4% lower than for unidirectional simulations. However, for unimodal sea conditions and for a sufficiently high level of nonlinearity (in our case this corresponds to shallower water), a large directional spreading results in an increase of the skewness. In this condition, the wave crests tend to be about 2.6% higher than the unidirectional case.

If bimodal seas are simulated, the increase of the angle $\delta$ between the independent spectra reduces significantly the vertical asymmetry of the wave profile. This produces crest heights which are lower than in an unimodal sea condition. For small $\delta$, however, the magnitude of the nonlinear asymmetry reaches relatively high values. Consequently, this yields a significantly different form of the wave crest distribution from the one related to unimodal sea states, as it lays outside the asymptotic 95% confidence interval of the latter.

The results of this study, based on the second-order model, are purely numerical and their value might therefore be limited by the underlying assumptions. In order to have a clear interpretation of them, further study on e.g. the coupling coefficient in the second-order contribution is needed. Furthermore, the analysis is only focused on time series. As the probability of any event in a point and in space may be quite different, it should be interesting in future to perform a similar analysis on spatial wave fields.

Acknowledgements

This work was carried out in the framework of the F.W.O. project G.0228.02 and G.0477.04., and the E.U. project SEAMOCS (contract MRTN-CT-2005-019374). The numerical simulations were performed by using the K.U. Leuven’s scientific computing facilities, including BeGrid (BELNET Grid Initiative). M.O. acknowledge Al Osborne for discussions during the early stages of this work. A.T. would also like to thank Patrick Willems for helpful comments and suggestions.
Appendix A. Interaction kernels

The positive and the negative interaction kernels in (9) have the following forms [5]:

\[ K_{ijlm}^+ = \left[ D_{ijlm}^+ - (k_i k_j \cos(\theta_l - \theta_m) - R_i R_j) \right] (R_i R_j)^{-1/2} + (R_i + R_j), \]

\[ K_{ijlm}^- = \left[ D_{ijlm}^- - (k_i k_j \cos(\theta_l - \theta_m) + R_i R_j) \right] (R_i R_j)^{-1/2} + (R_i + R_j), \]

where

\[ D_{ijlm}^+ = \frac{(\sqrt{R_i} + \sqrt{R_j})(\sqrt{R_i}(k_i^2 - R_i^2) + \sqrt{R_j}(k_j^2 - R_j^2))}{(\sqrt{R_i} + \sqrt{R_j})^2 - k_{ijlm}^+ \tanh(k_{ijlm}^+ h)} + 2(\sqrt{R_i} + \sqrt{R_j})^2 (k_i k_j \cos(\theta_l - \theta_m) - R_i R_j) \]

\[ D_{ijlm}^- = \frac{(\sqrt{R_i} - \sqrt{R_j})(\sqrt{R_i}(k_i^2 - R_i^2) - \sqrt{R_j}(k_j^2 - R_j^2))}{(\sqrt{R_i} - \sqrt{R_j})^2 - k_{ijlm}^- \tanh(k_{ijlm}^- h)} + 2(\sqrt{R_i} - \sqrt{R_j})^2 (k_i k_j \cos(\theta_l - \theta_m) + R_i R_j) \]

\[ k_{ijlm}^+ = \sqrt{k_i^2 + k_j^2 - 2k_i k_j \cos(\theta_l - \theta_m)}, \]

\[ k_{ijlm}^- = \sqrt{k_i^2 + k_j^2 + 2k_i k_j \cos(\theta_l - \theta_m)}, \]

\[ R_i = \omega_i^2 / g. \]

References


