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The nonlinear dynamics of rogue waves and holes in deep-water gravity wave trains

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Abstract

Rogue waves are rare “giant”, “freak”, “monster” or “steep wave” events in nonlinear deep water gravity waves which occasionally rise up to surprising heights above the background wave field. *Holes* are deep troughs which occur before and/or after the largest rogue crests. The dynamical behavior of these giant waves is here addressed as solutions of the nonlinear Schrödinger equation in both 1 + 1 and 2 + 1 dimensions. We discuss analytical results for 1 + 1 dimensions and demonstrate numerically, for certain sets of initial conditions, the ubiquitous occurrence of rogue waves and holes in 2 + 1 spatial dimensions. A typical wave field evidently consists of a background of *stable wave modes* punctuated by the intermittent upthrusting of *unstable rogue waves*. © 2000 Published by Elsevier Science B.V.

Keywords: Rogue waves; Freak waves; Benjamin–Feir instability; Nonlinear Schrödinger equation; Steep wave events; Sudden steep events

1. Introduction

Herein we study the phenomenon of “rogue” waves from the point of view of the simplest known *nonlinear* wave equations governing deep water wave trains: the *nonlinear Schrödinger equations* (NLS) in 1+1 (x,t) and 2+1 (x,y,t) dimensions. We seek to discover whether these equations and

their higher order extensions predict rogue-wave-like behavior. These equations are among the many new nonlinear wave equations that have been discovered over the past 30 years, many of which are integrable [1,2]. The principal tool for solving 1+1 NLS is the *inverse scattering transform* (IST), a generalization of linear Fourier analysis; 1+1 NLS was first integrated for *infinite-line boundary conditions* [3] ($\psi(x,t) \rightarrow 0$ as $|x| \rightarrow \infty$) and then for *periodic boundary conditions* [4] ($\psi(x,t) = \psi(x+L,t)$, L the period):

$$i(\psi_t + C_g \psi_x) - \mu \psi_{xx} - \nu \psi |\psi|^2 = 0 \quad (1)$$

where $C_g = \omega_0/2k_0$, $\mu = \omega_0/8k_0^2$ and $\nu = \omega_0 k_0^2/2$. The solutions of the 1+1 NLS equation, $\psi(x,t)$,

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describe the *complex envelope function* of a deep-water wave train whose *surface elevation*, $\eta(x,t)$, is given by $\eta(x,t) = \text{Re}[\psi(x,t)e^{ik_0x - \omega_0t}] + \text{Higher Order Terms}$ [5]; k_0 is the carrier wave number and ω_0 is its frequency; $\omega_0^2 = gk_0$ is the dispersion relation, g is the acceleration of gravity. The NLS equation has many important applications in modern physics, including plasmas [6], nonlinear optics [7], nonlinear control [8] and surface and internal water waves [1,2].

The 1+1 NLS equation describes the *nonlinear self-modulation* of a deep-water wave train which has a narrow-banded Fourier spectrum. The case for *infinite-line boundary conditions* has both *soliton* and *radiation modes* in analogy with the IST solution of the Korteweg–deVries (KdV) equation, where the soliton paradigm was first introduced [9]. The *wave group soliton solutions* of the NLS equation were identified early on as being an important and exotic physical manifestation of nonlinearity in deep water waves [5]. Later numerical and experimental work established the *lack of ubiquity* of solitons in the case for 2+1 dynamics (space, x , y , time, t) [1–5]. It is for this reason that the soliton paradigm was not pursued in much of the subsequent research in the field, particularly in 2+1 dimensions. This has left open the issue of the possible existence of other types of “coherent structures” in the 2+1 dimensional wave field, a major topic of discussion herein.

For the 1+1 NLS equation with *periodic boundary conditions* the situation is considerably more complex. Typically one considers initial conditions for which a “small-amplitude modulation” modifies a sine-wave carrier of amplitude a [10]. One finds that the solitons are replaced by *unstable modes* whose behavior is physically different from many points of view [12,13]. The unstable modes satisfy the classical definition given by linear instability analysis [10], i.e. for an unstable mode the modulation dispersion relation gives imaginary frequency, leading to the exponential growth of a small-amplitude modulation. From the point of view of the 1+1 NLS equation one can also address these modes in terms of “nonlinear instability analysis” (using the IST), with which one is able to provide not only the initial (possibly exponential) growth of the modulation, but also the entire space/time evolution. Herein we are tempted to call the NLS unstable modes

“rogue” or “giant” waves because of several of their properties [12,13]: (1) Rogue waves generally occur in rough sea conditions, appearing intermittently in space and time. (2) Their occurrence is not necessarily governed by the same probability laws which describe random, near gaussian waves. (3) In some sea states rogue waves are common, while in others they may not occur at all. (4) Rogues are often accompanied by “holes”, i.e. deep troughs occurring before/after the largest crest. Examples of these physical properties are illustrated below in the analytical and numerical examples.

2. Background on periodic IST in 1+1 dimensions

The unstable “rogue” modes of the 1+1 NLS Eq. (1) correspond to complex IST eigenvalues, $\lambda \pm \epsilon$ ($\epsilon \sim$ modulation amplitude), which are constants of the motion for the NLS equation [4]. The physical behavior of an unstable mode is governed by the Benjamin–Feir instability [4–6]. For most of its lifetime an unstable mode may not appear much different than the *stable modes* in the wave field. Occasionally, however, an unstable mode rises up (initially exponentially fast) to a height 3 or 4 times the height of the stable background wave field; after a brief moment of glory the unstable mode disappears into the background waves again. This process is approximately repeated in time through the mechanism of Fermi–Pasta–Ulam recurrence [4–6]. To clarify these ideas we briefly discuss the inverse scattering transform in 1+1 dimensions with periodic boundary conditions [4]. The NLS equation has the general IST spectral solution

$$\psi(x,t) = a\theta_N(x,t|\mathbf{B},\phi^-)/\theta_N(x,t|\mathbf{B},\phi^+)e^{2ia^2t} \quad (2)$$

where $\theta_N(x,t|\mathbf{B},\phi^\pm)$ are Riemann θ -functions with N degrees of freedom and a is the carrier wave amplitude. The ϕ^\pm are two sets of phases and \mathbf{B} is the “interaction” or “period” matrix which is $2N \times 2N$ for N an integer; both \mathbf{B} and ϕ^\pm are parameters determined by the inverse scattering transform. Diagonal elements of the matrix \mathbf{B} contain the “degrees of freedom”, off-diagonal elements describe the “interactions” among them. *Each unstable mode* has a 2×2 interaction matrix. For a small-amplitude modu-

lation of a plane wave, the simplest IST modes are of two types: (1) Eigenvalues on the real axis which lead to *stable* dynamics and (2) eigenvalues near the imaginary axis which lead to the *unstable* (rogue) modes. Consequently, it is straightforward to show analytically that the *spectral decomposition* for the 1+1 NLS equation can be written as a linear superposition of *stable modes* and *unstable modes*, plus *mutual nonlinear interactions*. This analysis is in complete analogy with a similar derivation for the shallow water KdV equation [9]. For the 1+1 NLS equation we have the following results [12]:

$$\ln \psi(x, t) = \ln \psi_{\text{stable}}(x, t) + \ln \psi_{\text{unstable}}(x, t) + \ln \psi_{\text{interactions}}(x, t) \quad (3)$$

The term ψ_{stable} (real axis modes) refers to waves which are stable with respect to the Benjamin–Feir instability, while ψ_{unstable} (imaginary axis modes)

refers to waves which are unstable. For the present work the utility of Eq. (3) rests not with the specific expressions for the terms on the right-hand-side of the equation; these lengthy derivations will be documented elsewhere [12]. The unstable modes rise up out of the background wave field and then subsequently disappear again, repeating the process quasi-periodically in time. Here is the fundamental picture that we have in mind for a *random oceanic wave field*: *Rare rogue waves occasionally rising up to immense heights out of a benign, near gaussian background in a periodic or quasi-periodic fashion*. Details for shallow-water random wave trains, using the inverse scattering transform, are given elsewhere [14]. Some of the results discussed herein relate to the generalization of this latter work to deep water.

We now give two analytical examples of unstable modes which are readily found from the periodic

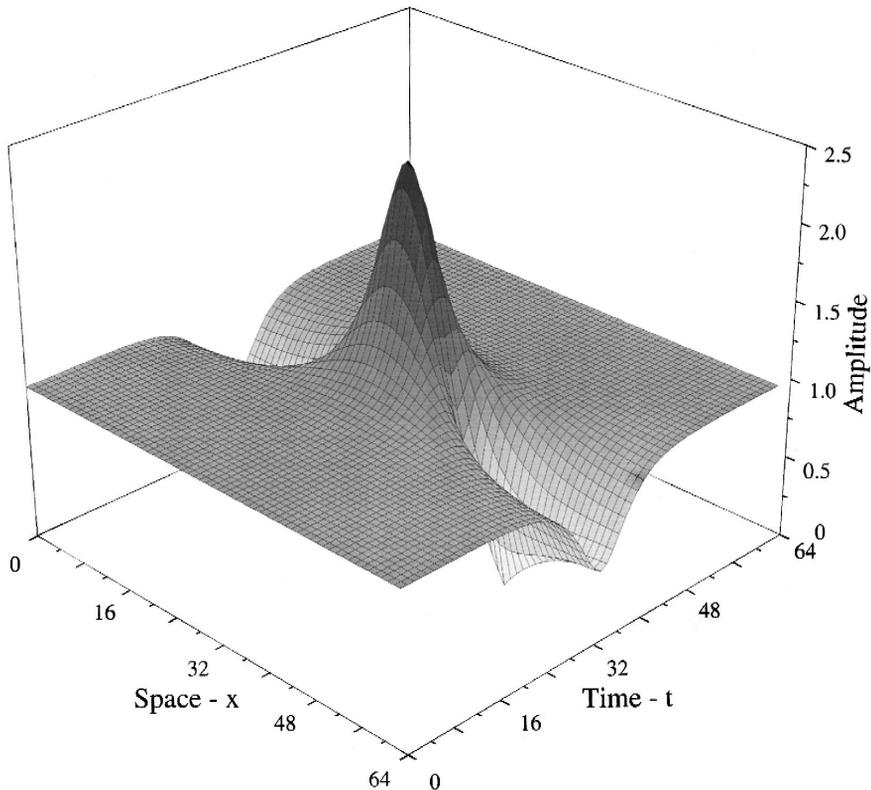


Fig. 1. Space–time evolution of the modulus, $|\psi(x, t)|$, of an unstable mode in 1+1 dimensions with IST eigenvalue $\lambda = ia2^{-1/2}$.

IST solution for the 1 + 1 NLS equation in adimensional form (use $u = \lambda\psi$, $X = x - C_g t$, $T = \mu t$, $\lambda = \sqrt{\nu/2\mu}$, in 1 to find $iu_T + u_{XX} + 2|u|^2 u = 0$) [15–18]. These two examples are particular homoclinic solutions (i.e. $\epsilon = 0$) for $\lambda = ia/\sqrt{2}$

$$u(X,T) = a \left[\frac{\cos[\sqrt{2} aX] \operatorname{sech}[\sqrt{2} a^2 T] + i\sqrt{2} \tanh[2a^2 T]}{\sqrt{2} - \cos[\sqrt{2} aX] \operatorname{sech}[\sqrt{2} a^2 T]} \right] e^{2ia^2 T} \tag{4}$$

and for $\lambda = ia\sqrt{2}$

$$u(X,T) = a \left[1 + \frac{2(\cos[4\sqrt{2} a^2 T] + i\sqrt{2} \sin[4\sqrt{2} a^2 T])}{\cos[4\sqrt{2} a^2 T] + \sqrt{2} \cosh[2aX]} \right] e^{2ia^2 T} \tag{5}$$

Both equations are seen to be modulations of the simple plane-wave solution of NLS, $u(X,T) = ae^{2ia^2 T}$. Eq. (4) is periodic in *space* and rises up *once* in its lifetime to its maximum value ($\sim 2.4a$) at $T = 0$, Fig. 1. Eq. (5) is periodic in *time* and rises up to its maximum value ($\sim 3.8a$) at $X = 0$, Fig. 2.

Eqs. (4,5) are simple examples of an infinite class of more general unstable solutions of 1+1 NLS [4]. The work of Yuen and Lake [10], in which unstable modes arising from small-amplitude sinusoidal modulations were systematically studied for a large number of numerical and experimental cases, is but a small subset of points near the imaginary axis interval, $i0 \leq \lambda \leq ia$, i.e. between the origin of the complex plane and the carrier wave at $\lambda = ia$. However, the periodic solutions of the NLS equation can generally lie *anywhere* in the complex plane [4]. This suggests that a large body of theoretical and experimental work remains to be conducted for deep-water wave dynamics in 1+1 dimensions.

3. Numerical simulations in 2+1 dimensions

We now address the case for 2+1 *dimensions*, where rogue wave dynamics are much less well

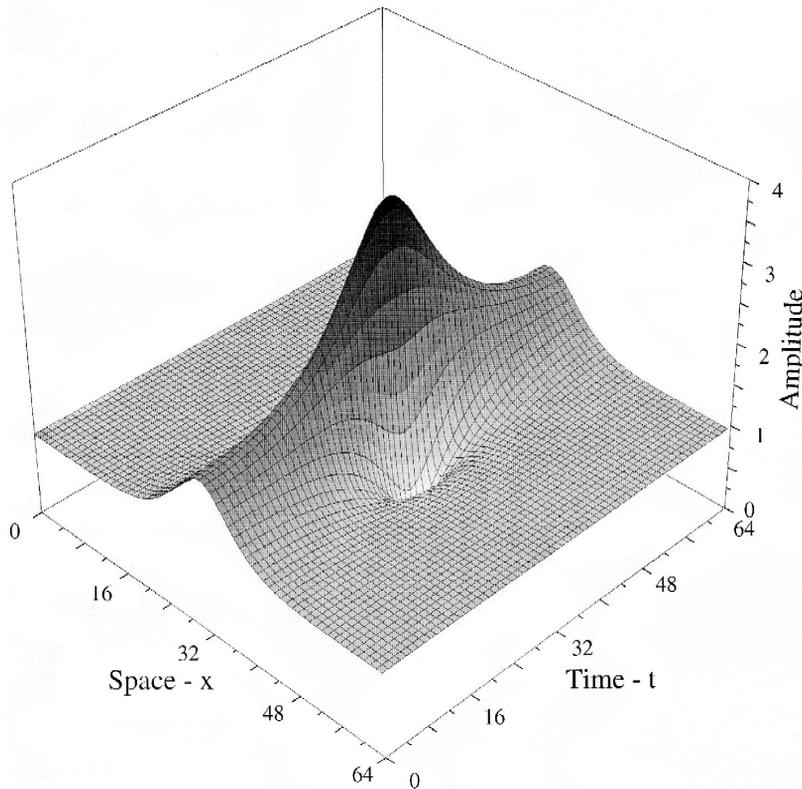


Fig. 2. Space–time evolution of the modulus, $|\psi(x,t)|$, of an unstable mode in 1+1 dimensions with IST eigenvalue $\lambda = ia2^{1/2}$.

understood. Here the 2+1 NLS equation is *not* exactly integrable and the power of the IST is not easily brought to bear to obtain analytical results for this difficult problem. Do rogue waves occur in the 2+1 problem and if so, are they larger/smaller than their 1+1 counterparts? Do other solutions exist which might contain new ingredients not seen in 1+1 dimensions? Can any of the properties of the IST solution in 1+1 carry over to the 2+1 problem? To answer some of these questions we have conducted extensive numerical simulations for the NLS equation in 2+1 dimensions:

$$i(\psi_t + C_g \psi_x) - \mu \psi_{xx} + \rho \psi_{yy} - \nu \psi |\psi|^2 = 0 \quad (6)$$

where $\rho = \omega_0/4k_0^2$. Previous studies have shown that knowledge of the behavior of a nonlinear wave equation in 1+1 dimensions may not tell us very

much about extensions to 2+1 dimensions [10]. It has been noticed by a number of observers that the y -component ($\rho \psi_{yy}$) of (6), arising from the transverse component of the Benjamin–Feir instability), can lead to the disintegration of envelope solitons: (1) Yuen and coworkers found that envelope solitons in 1+1 dimensions are no longer stable in 2+1 [10]. (2) Experiments by Hammack [1,2] indicate that the transverse (Benjamin–Feir) instability in 2+1 dimensions indeed leads to the destruction of solitons. (3) Theoretical work supports these results [19]. The very existence of “coherent structures” in 2+1 dimensions (or “unstable modes” in the sense of 1+1 dimensions) has therefore been left in doubt. A major aim of our paper is to address this area of research with new numerical results leading to the conclusion that unstable modes do indeed exist in

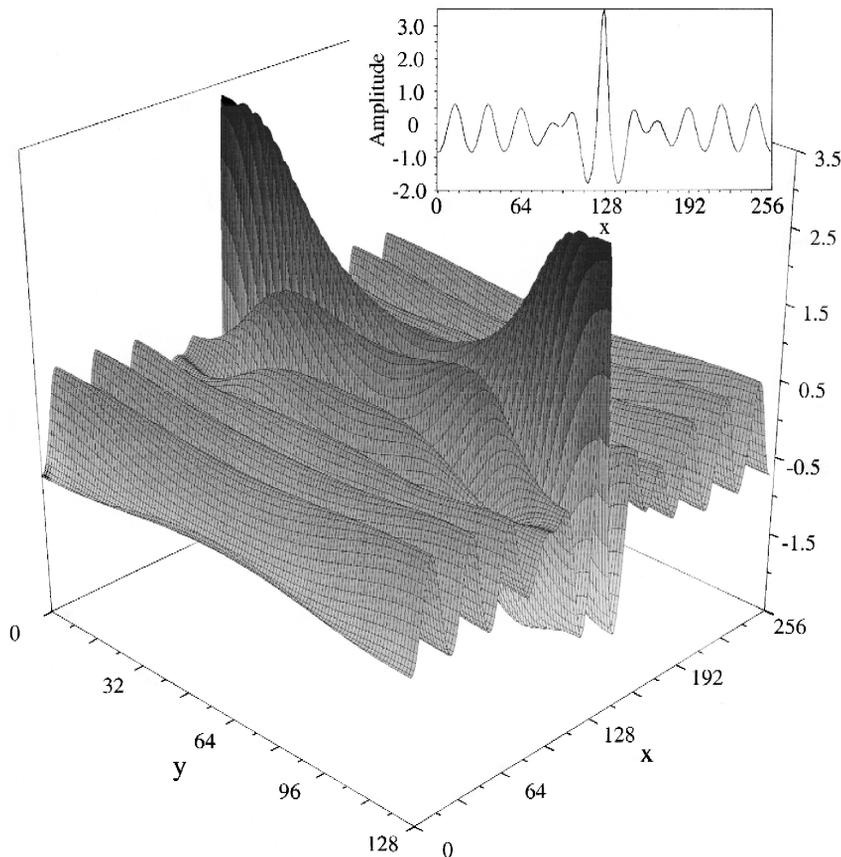


Fig. 3. Free surface elevation, $\eta(x, y, t_0)$, for a numerical solution to NLS in 2+1 dimensions. See text for details.

2+1, and that they can take the form of large amplitude “rogue” waves. One goal of the present work is to provide evidence that, for a wide range of initial conditions, “coherent structures” do indeed exist in the surface wave field and are ubiquitous in our simulations of deep water gravity waves. However, these structures are not strictly solitons, but are instead “unstable modes” or “rogue waves”.

Numerical simulations of the 2+1 NLS equation (our computer code is similar to that in Ref. [11]) are shown in Figs. 3, 4 and 5, where we graph the free surface elevation, $\eta(x, t)$, at the particular instant in time when a “rogue” mode rises to its maximum height; all the necessary Stokes corrections have been included in the free surface elevation [5]. In order to observe these solutions it has been necessary

to make high-resolution movies at a rapid frame rate. This has allowed us to capture the unstable modes very near their maximum amplitudes. We begin each of these simulations using a small-amplitude modulation in *both* the x and y directions, characterized by wave numbers K_x and K_y . We have chosen each pair (K_x, K_y) to lie within the unstable region of the linear instability diagram of the 2+1 NLS equation. We consider a carrier wave with wave number, k_0 , propagating in the x direction, perturbed by side-bands in the direction of propagation of the carrier wave and additional side-bands perpendicular to it. Typically we choose the amplitude of the perturbation to be ~ 0.1 times the carrier wave amplitude. The steepness of the carrier wave, $k_0 a$, was chosen to be 0.08, consistent with the physical derivation of

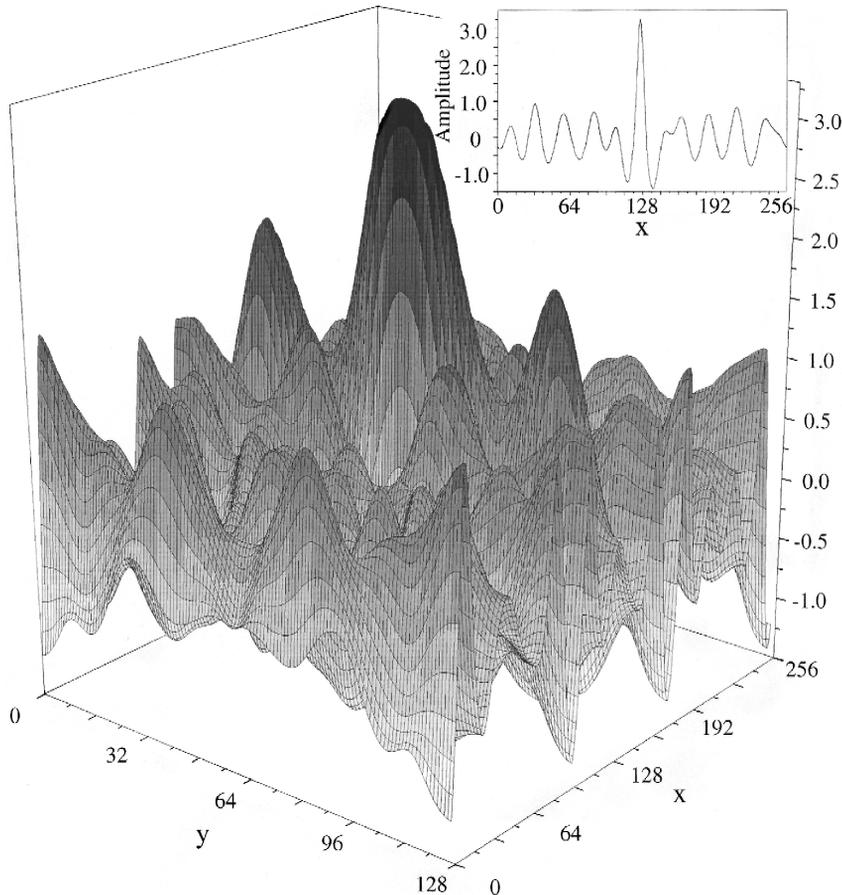


Fig. 4. Free surface elevation, $\eta(x, y, t_0)$, for a numerical solution to NLS in 2+1 dimensions. See text for details.

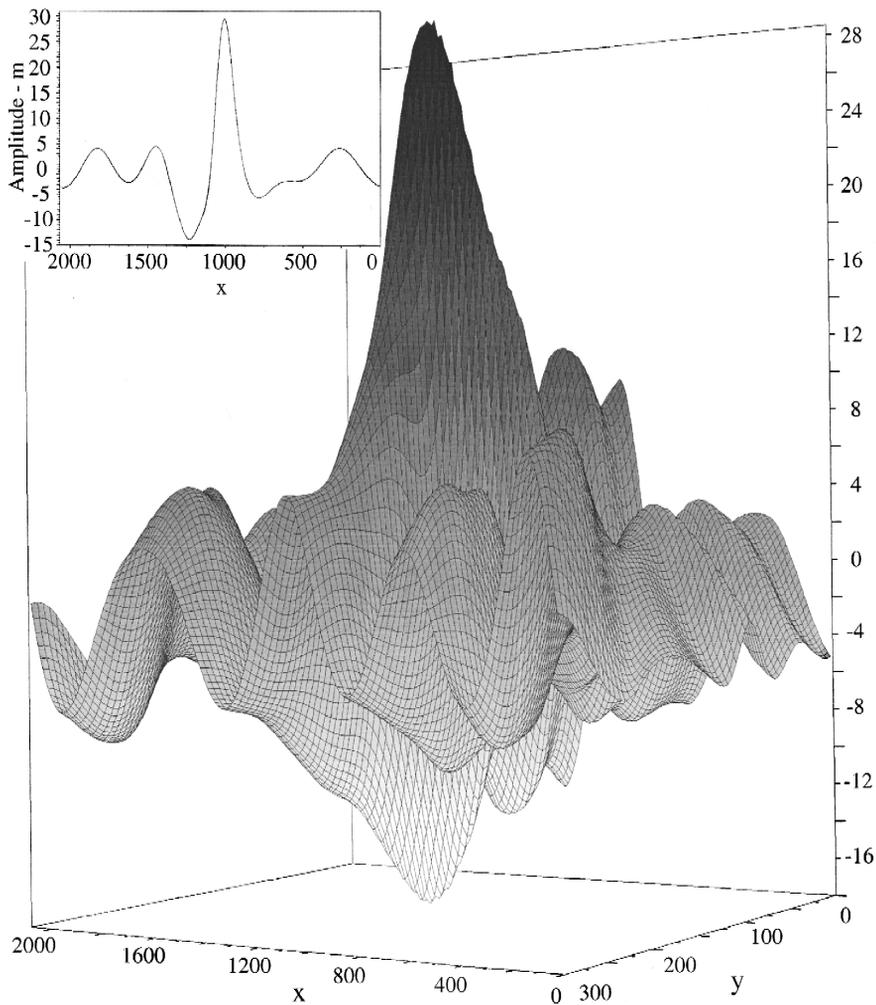


Fig. 5. A giant rogue wave in 2+1 dimensions in dimensional units of meters. The surface elevation, $\eta(x,t)$, is graphed as a function of space and time.

the NLS equation and with common oceanic conditions. The modulation wavelengths, $L_x = 2\pi/K_x$, $L_y = 2\pi/K_y$, are assumed to be much longer than the carrier wavelength, $L_0 = 2\pi/k_0$.

Fig. 3 is a case for which the transverse modulation has 10 times the wavelength of the modulation along the direction of propagation. This is initially a wave train which is almost unidirectional and subsequently evolves into a rogue wave, whose dynamics are influenced both by the parallel and transverse parts of the Benjamin–Feir instability. The motion first evolves as a relatively long-crested wave, which

subsequently loses its long-crestedness over time and leaps up out of the background sea state as a broad but finite-length wave front, localized in the x and y directions. Shown in the inset of Fig. 3 is a space series, $\eta(x, y=0, t_0)$, for $t=t_0$ occurring at the maximum rogue height of 2.6 carrier wave amplitudes. Fig. 4 shows an example for which the wave evolution is nearly isotropic and the result is a broad peak which rises up in the middle of relatively “random appearing” background waves. Here the inset indicates that the maximum rogue wave height is 2.4 times the carrier height.

In Fig. 5 we give an example for which the units are in meters so that we have a case which is oceanic in scale; the lateral modulation is 20 times longer than the modulation in the direction of propagation. We note that the background wave field (the carrier) has a height of 10 m, while the rogue wave has a height of 42 m. The maximum crest of 29 m is accompanied by a minimum or hole 13 m deep; the wave length of the rogue is ~ 450 m, corresponding to a maximum steepness of $k_0 a \sim 0.33$. To us these results are surprising not only because of the large ratio of maximum amplitude/carrier wave amplitude ~ 4.2 but also because of the large increase in steepness, from 0.08 to 0.33. These values exceed most of the cases we have studied in 1+1 dimensions and, while they do not reach the wave-breaking limit, they are nevertheless very nonlinear and warrant studies at higher order (see below). We have conducted several hundred simulations in 2+1 up to the present time and, details of these results will be given in subsequent publications. Nevertheless, we point out that the rogue waves in 2+1, for the initial conditions used by us, the unstable modes tend to be somewhat higher than their counterparts in 1+1 dimensions.

We have also studied this phenomenon from the point of view of the higher order NLS equation [20], including all of the necessary Stokes corrections in the free surface elevation. In both 1+1 and 2+1 dimensions we find that nonlinear unstable modes exist and that they have most of the properties of those at lower order. This suggests that by going to higher order and to higher dimensions, the existence of the unstable modes is a generic property of deep water wave trains. Our results are also supported by recent studies of unstable modes in the 1+1 Euler equations [21]. Continued exploration of the 2+1 NLS equation from numerical and theoretical points

of view, and its higher order extensions, therefore seems warranted.

The challenge now is to conduct a large body of experiments (laboratory and oceanic) to verify the theoretical and numerical scenario given herein for the occurrence of rogue waves in 1+1 and 2+1 dimensions to arbitrary order in nonlinearity.

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