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Relaxation of finite perturbations: Beyond the fluctuation-response relation

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We study the response of dynamical systems to finite amplitude perturbation. A generalized fluctuation-response relation is derived, which links the average relaxation toward equilibrium to the invariant measure of the system and points out the relevance of the amplitude of the initial perturbation. Numerical computations on systems with many characteristic times show the relevance of the above-mentioned relation in realistic cases. © *2003 American Institute of Physics.* [DOI: 10.1063/1.1579643]

Understanding the behavior of a dynamical system out of its equilibrium is a crucial issue of statistical physics. In the case of an infinitesimal perturbation that shifts the system out of equilibrium, the classical fluctuation-response theorem allows one to determine the linear response of the system in terms of its equilibrium properties, i.e., correlation functions. While the behavior of infinitesimal perturbations gives relevant information for problems of statistical mechanics, for climate and geophysical models the main goal is to characterize the relaxation of large perturbations, which cannot be obtained from the linear response theorem. We present here a generalization of the fluctuation-response relation, which holds for finite amplitude perturbations, providing a tool for extracting nonequilibrium behavior out of equilibrium features of the system. We also discuss the nontrivial role of the amplitude of perturbations in systems where many characteristic time scales are present.

I. INTRODUCTION

The fluctuation-response (F/R) relation has a deep relevance in statistical physics and more generally in systems with chaotic dynamics (in particular in hydrodynamics[1]). The relevance of a connection between ''nonequilibrium'' features (i.e., response to an external perturbation) and ''equilibrium'' properties (i.e., time correlations computed according to the invariant measure) is well known in statistical mechanics. We can mention the important Green–Kubo formulas in the linear response theory.[2] Beyond statistical physics, another field where the F/R problem has obvious relevance is climate research.[3] One of the key problems is the possibility to understand the response of the present climate to some violent changes (e.g., a volcanic eruption). The essential point is the possibility that the recovery of the climate system from a perturbation (response) can be estimated from its time history (correlation times of the unperturbed system).

Assuming that the system is mixing and has invariant probability density function (pdf) $\rho(\mathbf{x})$, it is possible to derive the following F/R relation. Let us denote by $\mathbf{x}(t) = (x_1(t),\ldots,x_N(t))$ the state of the system at time t. If at the initial time $t=0$ the system is perturbed by $\delta\mathbf{x}(0) = (\delta x_1(0),\ldots,\delta x_N(0))$, the average evolution of the perturbation $\langle \delta x_i(t)\rangle$ with respect the unperturbed trajectory is

$$\langle \delta x_i(t)\rangle = \sum_i R_{i,j}(t)\,\delta x_j(0), \tag{1}$$

where

$$R_{i,j}(t) = \left\langle \frac{\delta x_i(t)}{\delta x_j(0)} \right\rangle = \langle x_i(t) f_j(\mathbf{x}(0))\rangle \tag{2}$$

and the function f_j depends on $\rho(\mathbf{x})$ as

$$f_j(\mathbf{x}) = -\frac{\partial \ln \rho(\mathbf{x})}{\partial x_j}. \tag{3}$$

In Sec. II we will give a complete derivation of the above-mentioned formulas.

As far as we know, the F/R problem had been studied only for infinitesimal perturbations. For statistical mechanics problems it is relevant to deal with infinitesimal perturbations on the microscopic variables. In a similar way this problem has importance in many analytical approaches to the statistical description of hydrodynamics where Green functions are naturally involved both in perturbative theory and closure schemes.[1,4]

On the other hand in geophysical or climate problems the interest in infinitesimal perturbation seems to be rather academic, while the interesting problem is the behavior of relaxation of large fluctuations in the system due to fast changes of the parameters.

In this paper we want to address the problem of the F/R relation for noninfinitesimal perturbations. In Sec. II we will show that it is possible to generalize the F/R relation to large perturbations, involving rare events of the invariant measure. Section III is devoted to a discussion on the connections, and differences, between our approach and well-known results in dynamical system theory. In Sec. IV we will discuss the application to systems involving a single characteristic time, while Sec. V is devoted to systems with many characteristic times. Section VI is devoted to conclusions and the Appendix contains some technical remarks.

II. THEORETICAL BACKGROUND

In the following we will consider a dynamical system with evolution $\mathbf{x}(t)=\phi^t\mathbf{x}(0)$ of the N-dimensional vector $\mathbf{x}$. For generality, we will explicitly consider the case in which the time evolution can also be not completely deterministic (e.g., stochastic differential equations). We will assume the existence of an invariant probability distribution $\rho(\mathbf{x})$ and the ergodicity of the system so that

$$\langle A\rangle \equiv \lim_{T\to\infty}\frac{1}{T}\int_0^T A(\mathbf{x}(t))\,dt=\mu(A)\equiv\int A(\mathbf{x})\rho(\mathbf{x})\,d\mathbf{x} \tag{4}$$

for any (smooth enough) observable A.

Our aim is the understanding of the mean response $\langle\delta A(t)\rangle$ of a generic observable A initially perturbed with $\delta A(0)$. The first step is the study of one component of $\mathbf{x}$, i.e., $\langle\delta x_i(t)\rangle$ with an initial (nonrandom) perturbation $\delta\mathbf{x}(0)=\delta\mathbf{x}_0$. Introducing the probability of transition from $(\mathbf{x}_0,0)$ to $(\mathbf{x},t)$, $W(\mathbf{x}_0,0\to\mathbf{x},t)$ [for a deterministic system we have $W(\mathbf{x}_0,0\to\mathbf{x},t)=\delta(\mathbf{x}(t)-\phi^t\mathbf{x}_0)$], we can easily write an expression for the mean value of the variable computed along the perturbed trajectory $x_i'(t)=x_i(t)+\delta x_i(t)$:

$$\langle x_i'(t)\rangle=\int\int x_i\rho'(\mathbf{x}_0)W(\mathbf{x}_0,0\to\mathbf{x},t)\,d\mathbf{x}\,d\mathbf{x}_0, \tag{5}$$

where $\rho'(\mathbf{x})$ is the initial distribution of perturbed system, which is related to the invariant distribution by $\rho'(\mathbf{x}_0)=\rho(\mathbf{x}_0-\delta\mathbf{x}_0)$. Noting that the mean value of $x_i(t)$ can be written in a similar way;

$$\langle x_i(t)\rangle=\int\int x_i\rho(\mathbf{x}_0)W(\mathbf{x}_0,0\to\mathbf{x},t)\,d\mathbf{x}\,d\mathbf{x}_0, \tag{6}$$

one has

$$\begin{aligned}\langle\delta x_i(t)\rangle&=\int\int x_i\frac{\rho(\mathbf{x}_0-\delta\mathbf{x}_0)-\rho(\mathbf{x}_0)}{\rho(\mathbf{x}_0)}\rho(\mathbf{x}_0)\\&\quad\times W(\mathbf{x}_0,0\to\mathbf{x},t)\,d\mathbf{x}\,d\mathbf{x}_0\\&=\langle x_i(t)F(\mathbf{x}_0,\delta\mathbf{x}_0)\rangle,\end{aligned} \tag{7}$$

where

$$F(\mathbf{x}_0,\delta\mathbf{x}_0)=\left[\frac{\rho(\mathbf{x}_0-\delta\mathbf{x}_0)-\rho(\mathbf{x}_0)}{\rho(\mathbf{x}_0)}\right]. \tag{8}$$

For an infinitesimal perturbation $\delta\mathbf{x}(0)=(\delta x_1(0)\cdots\delta x_N(0))$ expanding Eq. (8) to first order one ends with the expression

$$\begin{aligned}\langle\delta x_i(t)\rangle&=-\sum_j\left\langle x_i(t)\left.\frac{\partial\ln\rho(\mathbf{x})}{\partial x_j}\right|_{t=0}\right\rangle\delta x_j(0)\\&\equiv\sum_j R_{i,j}(t)\,\delta x_j(0),\end{aligned} \tag{9}$$

which defines the linear response

$$R_{i,j}(t)=-\left\langle x_i(t)\left.\frac{\partial\ln\rho(\mathbf{x})}{\partial x_j}\right|_{t=0}\right\rangle \tag{10}$$

of the variable x_i with respect to a perturbation of x_j. Relation (10) is the generalization for non-Hamiltonian systems of the well-known fluctuation/response (F/R) relation.[2]

Let us note that in the general case the invariant measure $\rho(\mathbf{x})$ is not known, so Eq. (10) gives just qualitative information. In the case of Gaussian distribution, $\rho(\mathbf{x})$ factorizes and the linear response recovers the correlator

$$R_{i,j}(t)=\frac{\langle x_i(t)x_j(0)\rangle-\langle x_i\rangle\langle x_j\rangle}{\langle x_jx_j\rangle-\langle x_j\rangle\langle x_j\rangle}. \tag{11}$$

In the case of finite perturbations, the F/R relation (7) is typically nonlinear in the perturbation $\delta\mathbf{x}_0$ and thus no simple relations analogous to Eq. (10) exist. Nevertheless we can disentangle the different contributions in the response (7) by studying an initial perturbation whose only nonzero component is the jth one,

$$\delta^{(j)}\mathbf{x}(0)=(0,\ldots,0,\delta x_j(0),0,\ldots,0). \tag{12}$$

We therefore generalize the F/R relation (10) to nonlinear response of x_i to a perturbation on the j variable as

$$R_{i,j}(t)=\langle x_i(t)f_j(0)\rangle, \tag{13}$$

where f_j is given by

$$f_j(\mathbf{x}_0)=\frac{\rho(\mathbf{x}_0-\delta^{(j)}\mathbf{x}(0))-\rho(\mathbf{x}_0)}{\rho(\mathbf{x}_0)\,\delta x_j(0)}. \tag{14}$$

The explicit prediction of the response from Eq. (13) requires the analytic expression of the invariant pdf, which is in general not known. Nevertheless Eq. (7) guarantees the existence of a link between equilibrium properties of the system and the response to finite perturbations. This fact has a relevant consequence for systems with one single characteristic time: a generic correlation [e.g., the correlation (11)] in principle gives information on the relaxation time of finite size perturbations, even when the invariant measure ρ is not known.[5]

III. REMARKS ON THE CONNECTIONS BETWEEN F/R RELATION, DYNAMICAL SYSTEM THEORY, AND STATISTICAL MECHANICS

Since the F/R relation involves the evolution of differences between variables computed on two different realizations of the system, it is natural to conclude that this issue is closely related to the predictability problem and, more in general, to chaotic behavior. Actually, a detailed analysis shows that the two problems, i.e., F/R relation and predictability, have only a very weak connection. For the sake of completeness, we briefly discuss here the analogies and differences between these two issues.

The typical problem for the characterization of predictability is the evolution of the trajectory difference $\delta\mathbf{x}(t)$, in particular of $\langle \ln|\delta\mathbf{x}(t)|\rangle$ which defines the leading Lyapunov exponent λ. For small $|\delta\mathbf{x}(0)|$ and large enough t one has

$$\langle \ln|\delta\mathbf{x}(t)|\rangle \simeq \ln|\delta\mathbf{x}(0)|\lambda t. \tag{15}$$

On the other hand, in F/R issue one deals with averages of quantities with sign, such as $\langle \delta\mathbf{x}(t)\rangle$. This apparently marginal difference is very important and it is at the basis of the famous objection by van Kampen related to the standard derivation of the linear response theory.[6] In a nutshell, using the modern dynamical systems terminology, van Kampen's argument is as follows. Since in presence of chaos $|\delta\mathbf{x}(t)|$ grows exponentially in time, it is not possible to linearize Eq. (8) for time larger than $(1/\lambda)\ln(\Delta/|\delta\mathbf{x}(0)|)$, where Δ is the typical fluctuation of the variable $\mathbf{x}$. As a consequence, the linear response theory is expected to be valid only for extremely small and unphysical perturbations, in clear disagreement with the experience. A solution of this apparent paradox was proposed by Kubo who suggested that "*instability [of the trajectories] instead favors the stability of distribution functions, working as the cause of the mixing*."[7] More recent works have demonstrated the constructive role of chaos in F/R relation and the nonrelevance of van Kampen's criticism.[8,9] The objection by van Kampen remains nevertheless relevant for numerical computations of F/R relation (see the Appendix).

Fluctuation/response relation was developed in the context of statistical mechanics of Hamiltonian systems, but it also holds for nonconservative systems, and even nondeterministic systems (e.g., Langevin equations) and has no general relation with "chaotic quantities" such as Lyapunov exponents or Kolmogorov–Sinai entropy. This generated in the past some confusion about the applicability of F/R relation. For example, some authors claimed (with qualitative arguments) that in fully developed turbulence there is no relation between equilibrium fluctuations and relaxation to equilibrium[10] while the correct statement concerns the nonvalidity of the simplified relation (12) which holds only for systems with Gaussian statistics.

Thanks to its general validity and robustness, the F/R relation has also been used to obtain information on the unknown invariant measure $\rho(\mathbf{x})$ on the basis of the linear response $R_{i,j}(t)$. An important example comes from the field of disordered systems where the F/R had been applied to the study of aging phenomena.[11]

Concluding this short discussion on the connections between F/R relation, dynamical system theory, and statistical mechanics, we mention recent results about rigorous derivation of the Onsager reciprocity relations[12] and the macroscopic fluctuation theory for stationary nonequilibrium states[13] in a class of stochastic models describing interacting particles systems.

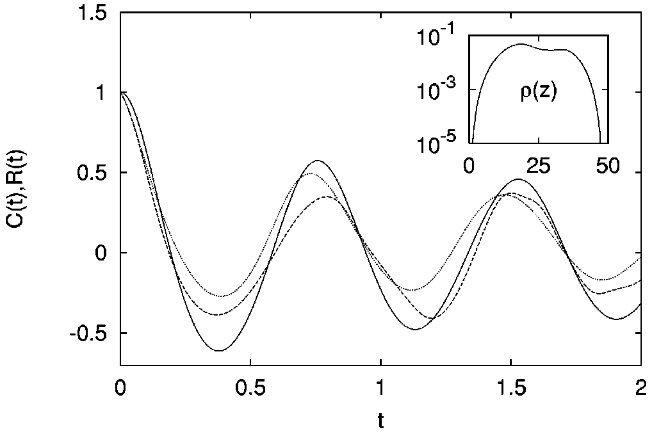

FIG. 1. Correlation function of the z variable of Lorenz model (solid line) compared with the mean response to different perturbations of the same variable. $\delta z_0 = 10^{-2}\sigma$ (dashed line), $\delta z_0 = \sigma$ (dotted line), with $\sigma = \sqrt{\langle z^2\rangle - \langle z\rangle^2} = 8.67$.

IV. SYSTEMS WITH A SINGLE CHARACTERISTIC TIME

Let us start by studying two examples of systems with a single characteristic time: a deterministic chaotic system (the Lorenz model) and a nonlinear Langevin process.

We first consider the Lorenz model[14]

$$\frac{dx}{dt} = \sigma(y-x),$$
$$\frac{dy}{dt} = -xz + rx - y, \tag{16}$$
$$\frac{dz}{dt} = xy - bz$$

with standard parameters for chaotic behavior: $b=8/3$, $\sigma=10$, and $r=28$. The correlation function (11) for the variable z, shown in Fig. 1, qualitatively reproduces the behavior of the response to different sizes of the perturbation of the z variable, ranging from infinitesimal ones up to the size of the attractor. The accuracy does not increase when decreasing the perturbation because the invariant distribution is not Gaussian (see Fig. 1) and thus the general correlation (10) should be used. We observe that the use of Eq. (10) instead of Eq. (11) is in general much more difficult because the invariant distribution is in general nonfactorable.

To better illustrate this point, let us now consider a system whose invariant probability distribution is known. In this case we can quantitatively compare the differences between the responses to infinitesimal and finite perturbations. Our example is provided by the stochastic process $x(t)$ determined by

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + \sqrt{2D}\xi(t), \tag{17}$$

where $\xi(t)$ is a white noise, i.e., a Gaussian process with $\langle \xi(t)\rangle = 0$ and $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$. The invariant probability distribution is[15]

$$\rho(x) = \mathcal{N} e^{-U(x)/D}, \tag{18}$$

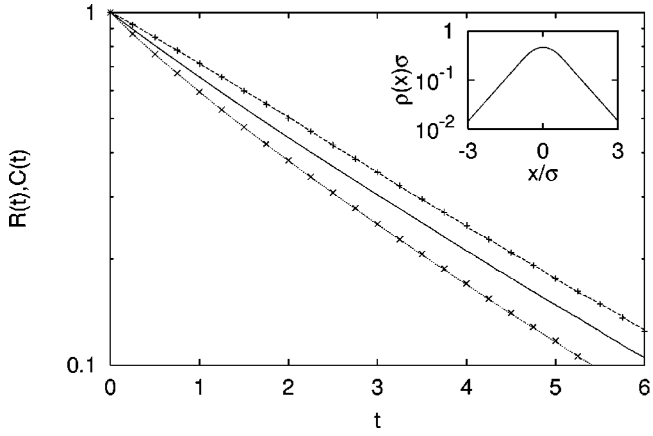

FIG. 2. Mean response of the stochastic differential equation $dx/dt = -B(x) + \sqrt{2D}\xi(t)$, with $D=1$, $B(x)=x$ for $|x|<1$ and $B(x)=1$ for $|x|>1$, to different perturbations: large $\delta x_0 = 2.3\sigma$ (+) and infinitesimal $\delta x_0 = 7.6\times 10^{-3}\sigma$ (×). In both cases the mean response is exactly predicted by the correlator $\langle x(t) f(\mathbf{x}(0))\rangle$ (dashed line for $\delta x_0 = 2.3\sigma$ and dotted line for $\delta x_0 = 7.6\times 10^{-3}$) according to Eq. (13) while the simple correlation $\langle x(t)x(0)\rangle/\sigma^2$ (solid line) just gives an estimate of the relaxation time. In the inset we show the invariant probability distribution $\rho(x)\sigma$ vs x/σ with $\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = 1.32$. Statistics is over 10^6 independent runs.

where $\mathcal{N}$ is fixed by normalization.

A Gaussian pdf is obtained using $U(x) = x^2/2$, which corresponds to the linear Ornstein–Uhlenbeck process $dx/dt = -x + \sqrt{2D}\xi(t)$. Our example uses a modified version of the Gaussian case,

$$U = \begin{cases} \frac{1}{2}x^2, & |x|<1 \\ |x| - \frac{1}{2}, & |x|>1. \end{cases} \tag{19}$$

The resulting pdf, shown in the inset of Fig. 2, has a Gaussian core, with exponential tails. Figure 2 also shows the response function for an infinitesimal and for a finite size perturbation. For both perturbations, the response function measured from the perturbed trajectories is exactly predicted by statistics of the unperturbed system according to Eq. (13), while the Gaussian correlation $C(t) = \langle x(t)x(0)\rangle/\sigma^2$ gives only an estimate of the relaxation time. By construction, the pdf of this system has larger tails than in the Gaussian case, thus large fluctuations decay slower than small ones. In the linear case the mean response is simply $R(t) = \exp(-t)$ and does not depend on the amplitude of the initial perturbation $\delta x(0)$.

The results obtained for the Lorenz model and for the nonlinear Langevin equations suggest that if only one characteristic time is present, the existence of the F/R relation allows for some qualitative results even in the absence of precise knowledge of ρ, both for infinitesimal and finite perturbation.

V. SYSTEMS WITH MANY CHARACTERISTIC TIMES

In systems with many characteristic times, different correlation functions do not show the same behavior, i.e., depending on the observable one can observe very different time scales, corresponding to the different decay times of the correlation functions $C_{j,j} = \langle x_j(t) x_j(0)\rangle$.[5] In addition, at variance with systems with one single time scale, here the amplitude of the perturbation can play a major role in determining the response, because different amplitudes may affect features with different time properties.

The link between equilibrium and relaxation properties established by the F/R relation (13) suggests that it is possible to relate different relaxation rates with the time scales measured by means of correlations. Consider the case of an observable A which depends on all the variables of the system $\{x_1,\ldots,x_N\}$. For infinitesimal perturbations, a straightforward generalization of Eqs. (1) and (2) gives

$$\langle \delta A(t)\rangle = \sum \langle A(\mathbf{x}(t)) f_j(\mathbf{x}(0))\rangle \delta x_j(0). \tag{20}$$

In the case of finite perturbations, as stressed in Sec. II, it is possible to write a F/R relation:

$$\langle \delta A(t)\rangle = \langle A(\mathbf{x}(t)) F(\mathbf{x}(0), \delta\mathbf{x}(0))\rangle \tag{21}$$

in which, at variance with Eq. (20), all the variables are mixed. In Eq. (21) the relaxation properties depend explicitly on the initial perturbation $\delta\mathbf{x}(0)$.

Depending on the choice of $A(\mathbf{x})$, different perturbations on A correspond to different amplitudes of the perturbations on each variable x_j. Consequently, one can think that it is possible to associate each perturbation to a certain subset of variables which are mainly perturbed. The relaxation of $\langle \delta A(t)\rangle$ will be ruled by the characteristic time of that particular subset.

In order to illustrate this issue we consider a shell model for turbulence.[16] Shell models are a simplified model for turbulent energy cascade, which describe the dynamics of velocity fluctuations at a certain scale $\ell_n = k_n^{-1}$ with a single shell-variable u_n. Wave numbers k_n are geometrically spaced as $k_n = k_0\lambda^n$, allowing one to cover a large range of scales with relatively few variables. A quadratic interaction between neighbor shell reproduces the main features of three-dimensional turbulence. The specific model we will use is

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n = i[k_{n+1}u_{n+1}^* u_{n+2} - \epsilon k_n u_{n+1} u_{n-1}^* + (1-\epsilon)k_{n-1}u_{n-2}u_{n-1}] + f_n, \tag{22}$$

where ν is the molecular viscosity, f_n is an external forcing which injects energy at large scale, and ϵ is a free parameter. In order to have the correct conservation laws (energy and helicity) in the inviscid unforced case one has to fix $\epsilon = 1/2$. The observable considered is the total energy $E(t) = \frac{1}{2}\Sigma_{n=1}^{N}|u_n(t)|^2$ which is the conserved quantity in the inviscid, unforced limit.[16]

In order to study the response to perturbations with different amplitude on E, we consider the following perturbed systems labeled with $i=1,\ldots,N$: $u_n^{(i)}(t) = u_n(t) + \delta u_n^{(i)}(t)$ where the initial perturbations $\delta u_n^{(i)}(0)$ are set in the following way:

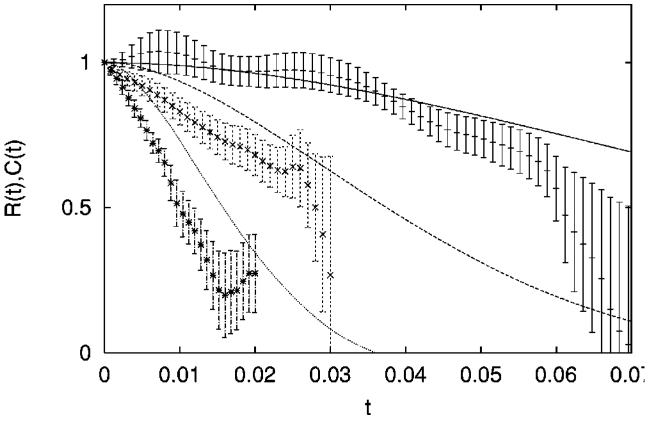

FIG. 3. Mean response $R(t)=\langle \delta E(t)/\delta E(0)\rangle$ of the total energy $E(t)=1/2\Sigma|u_n|^2$ of the shell model (22) to different amplitude perturbations: $\delta E(0)=5.5\times 10^{-3}$ (+), $\delta E(0)=1.7\times 10^{-3}$ (×), $\delta E(0)=4.5\times 10^{-4}$ (*). Varying the amplitude of the initial perturbation different relaxation rates are observed, and the response function is roughly similar to the correlation function of the corresponding largest perturbed shell: shell $n=12$ (solid line), shell $n=14$ (dashed line), shell $n=16$ (dotted line).

$$\delta u_n^{(i)}(0)=\begin{cases} 0, & 1\leqslant n\leqslant i-1 \\ \sqrt{\langle |u_n|^2\rangle} & i\leqslant n\leqslant N. \end{cases} \tag{23}$$

This corresponds to a set of initial perturbations of the energy

$$\langle \delta E_i(0)\rangle = \frac{1}{2}\sum_{n=i}^{N} \langle |u_n|^2\rangle. \tag{24}$$

Such a perturbation is motivated by the fact that in the unperturbed system the energy is distributed among the shells according to the Kolmogorov scaling $\langle |u_n|^2\rangle \sim k_n^{-2/3}$, and the smaller scales give smaller contributions to the energy $E(t)$. Thus it is natural to assume that a small perturbation of the energy will affect mainly the small scales.

For each perturbation δE_i, the average response of energy

$$\left\langle \frac{\delta E_i(t)}{\delta E_i(0)} \right\rangle = \left\langle \frac{\Sigma_{n=1}^{N} |u_n^{(i)}(t)|^2 - |u_n(t)|^2}{\Sigma_{n=1}^{N} |u_n^{(i)}(0)|^2 - |u_n(0)|^2} \right\rangle \tag{25}$$

reveals a close relation with the time correlation of the corresponding largest perturbed shell $u_i(t)$, as shown in Fig. 3. A measure of the relaxation time can be provided by the halving times $T_{1/2}$ of the mean response, at which $\langle \delta E_i(T_{1/2})\rangle = 1/2 \langle \delta E_i(0)\rangle$. The dependence of response times on the amplitude of the initial perturbation, shown in Fig. 4, reflects Kolmogorov scaling for characteristic times $\tau_n \sim k_n^{-2/3} \sim u_n^2 \sim \delta E_n$,

$$T_{1/2} \sim \delta E. \tag{26}$$

The above-mentioned results on the shell model show that the response to a finite size perturbation of a system with many characteristic times may depend on the amplitude of the perturbation. Thanks to the existence of F/R relation it is possible to establish a link between relaxation times of different perturbation and characteristic times of the system.

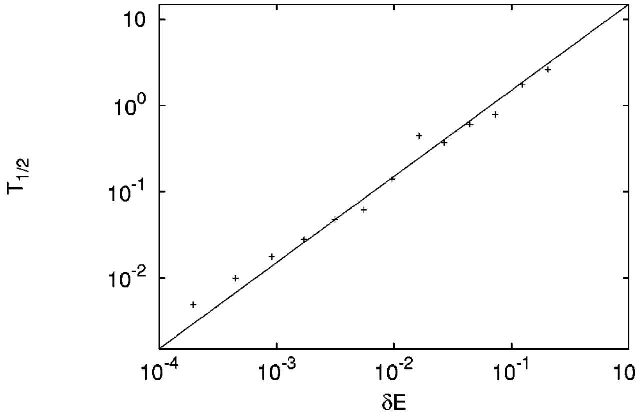

FIG. 4. Halving times $T_{1/2}$ of the mean response to different amplitude perturbations of the total energy $E(t)=1/2\Sigma|u_n|^2$ of the shell model (22): $R(t)=\langle \delta E(t)/\delta E(0)\rangle$. Solid line represents the dimensional scaling $T_{1/2} \sim \delta E$.

VI. CONCLUSIONS

Starting from the seminal works of Leith,[3,17] who proposed the use of F/R relation for understanding the response of the climatic system to changes in the external forcing, many authors tried to apply this relation to different geophysical problems, ranging from simplified models,[18] to general circulation models[19,20] and to the covariance of satellite radiance spectra.[21] Most applications have not taken into account the limits of applicability of the F/R relation, which has been used as a kind of approximation. We have shown that a F/R relation holds under very general conditions. The derivation in Sec. II clearly shows the limits of applicability in its simplest form [i.e., the Gaussian approximation (11)].

Our main result is the demonstration that an exact fluctuation/response relation holds also for noninfinitesimal perturbation. This relation involves the detailed form of the invariant probability distribution. In particular, in order to predict the mean response to large perturbations, one needs precise knowledge of the tails of the pdf.

We believe that this generalization of the usual linear response theory can be relevant in many applications. As an example, we can mention climate research, where our results imply the possibility, at least in principle, to understand the behavior of the system after a large impulsive perturbation (e.g., a volcanic eruption) in terms of the knowledge obtained from its time history. Of course one has to take into account the strong limitations due to the need to have a good statistics of rare events.

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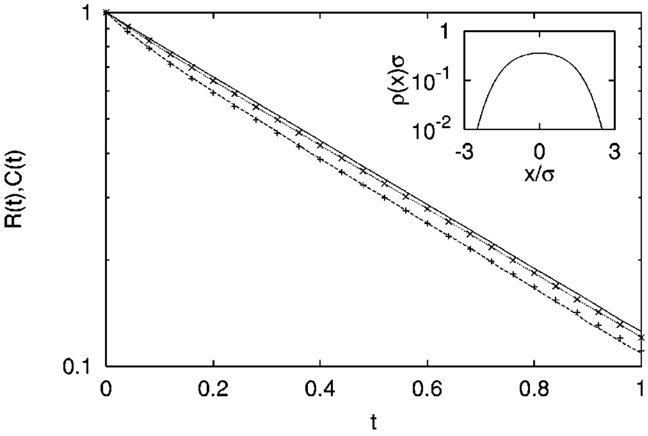

FIG. 5. Mean response of the stochastic differential equation $dx/dt = -B(x) + \sqrt{2D}\xi(t)$, with $D=1$, $B(x)=x+x^3$, to different perturbations: finite $\delta x_0 = 1.5\sigma$ (+) and infinitesimal $\delta x_0 = 1.5\times 10^{-2}\sigma$ (×). The mean response is exactly predicted by the correlator $\langle x(t) f(\mathbf{x}(0))\rangle$ (dashed line for $\delta x_0 = 1.5\sigma$ and dotted line for $\delta x_0 = 1.5\times 10^{-2}\sigma$) according to Eq. (13), while the simple correlation $\langle x(t)x(0)\rangle/\sigma^2$ (solid line) just gives an estimate of the relaxation time. In the inset we show the invariant probability distribution $\rho(x)\sigma$ vs x/σ with $\sigma = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = 0.68$. Statistics is over 10^6 independent runs.

APPENDIX: TECHNICAL DETAILS

In this appendix we want to discuss how van Kampen criticism is relevant for the numerical evaluation of infinitesimal response function.

In numerical simulations, $R_{i,j}(t)$ is computed perturbing the variable x_i at time $t=t_0$ with a small perturbation of amplitude $\delta x_i(0)$ and then evaluating the separation $\delta x_i(t)$ between the two trajectories $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ which are integrated up to a prescribed time $t_1 = t_0 + \Delta t$. At time $t=t_1$ the variable x_i of the reference trajectory is again perturbed with the same $\delta x_i(0)$, and a new sample $\delta\mathbf{x}(t)$ is computed and so forth. The procedure is repeated $M \gg 1$ times and the mean response is then evaluated according to Eq. (11).

In presence of chaos, the two trajectories $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ typically separate exponentially in time and the perturbed system relaxes to the unperturbed one only in average, therefore the mean response is the result of a delicate balance of terms which grow in time in different directions. The average error in the computation of $R_{i,j}(t)$ typically increases in time as $e^{L(2)t/2}/\sqrt{M}$, where $L(2)$ is the generalized Lyapunov exponent.[16] Thus very high statistics is needed in order to compute $R_{i,j}(t)$ for large t.[8]

We remark that the exponential growth is generally valid only for infinitesimal perturbation. When the perturbation reaches the typical size of the system, the difference between the perturbed and the unperturbed trajectory tends to saturate. Thus, for finite amplitude perturbations the mean response is the average of terms that remain of order $O(1)$, and less statistics is required to obtain convergence. In this sense the mean response to finite perturbation is more representative of the behavior of a single perturbation than in the infinitesimal case.

On the other hand, even if Eq. (13) is formally valid for arbitrary large perturbations, for practical use an upper limit exists due to finiteness of statistics. To predict the relaxation of a perturbation $\delta\mathbf{x}(0)$, one needs sufficient statistics for the convergence of $\rho(\mathbf{x}(0) - \delta\mathbf{x}(0))$. This request is more severe in systems where large fluctuations are suppressed. An example is provided by the stochastic model (17) with

$$U(x) = \tfrac{1}{2}x^2 + \tfrac{1}{4}x^4. \tag{A1}$$

Here the pdf has sub-Gaussian tails, and we observe the opposite behavior of the system (19), as shown in Fig. 5. While in the case with exponential tails we have a good statistical convergence for a perturbation greater than 2σ in the second system this perturbation is too large to obtain convergence even with huge statistics (10^9 runs).

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