Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

Scaling regimes of 2d turbulence with power-law stirring: theories versus numerical experiments

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Received 21 July 2009
Accepted 1 October 2009
Published 19 October 2009

Online at stacks.iop.org/JSTAT/2009/P10012
doi:10.1088/1742-5468/2009/10/P10012

Abstract. We inquire about the statistical properties of the pair formed by the Navier–Stokes equation for an incompressible velocity field and the advection–diffusion equation for a scalar field transported in the same flow in *two dimensions* (2d). The system is in a regime of fully developed turbulence stirred by forcing fields with Gaussian statistics, white noise in time and self-similar in space. In this setting and if the stirring is concentrated at small spatial scales, as if due to thermal fluctuations, it is possible to carry out a first-principles ultraviolet renormalization group analysis of the scaling behavior of the model. Kraichnan's phenomenological theory of two-dimensional turbulence upholds the existence of an inertial range characterized by inverse energy transfer at scales larger than the stirring one. For our model Kraichnan's theory, however, implies scaling predictions radically discordant from the renormalization group results. We perform accurate numerical experiments to assess the actual statistical properties of 2d turbulence with power-law stirring. Our results clearly indicate that an adapted version of Kraichnan's theory is consistent with the observed phenomenology. We also provide some theoretical scenarios to account for the discrepancy between renormalization group analysis and the observed phenomenology.

1742-5468/09/P10012+19$30.00

Keywords: critical exponents and amplitudes (theory), renormalization group, turbulence, stochastic processes (theory)

Contents

1. Introduction

Two-dimensional (2d) turbulence is interesting for several reasons. In laboratory experiments 2d turbulence has been realized and studied with electromagnetically driven liquid metals [60, 59, 14] and thin soap films [61]–[63]. In geophysical flows, vertical confinement suggests the possibility to describe the mesoscale dynamics of atmosphere and oceans in terms of two-dimensional fluid models [19, 23]. Indeed, observational data such as the Nastrom–Gage spectrum [55, 56], studies based on the MOZAIC database [53] and on the EOLE Lagrangian balloons in the low stratosphere [42] support the existence of a mesoscale $-5/3$ power-law energy spectrum which may be the consequence of a two-dimensional inverse cascade. Although recent studies [44, 47] suggest the occurrence of a fairly more complex physical phenomenology (see, e.g., [67, 68]), the 2d approximation remains an important benchmark for understanding the atmospheric physics at synoptics and planetary scales [45] as well as in other geophysical contexts (see [65, 40] and references therein). For example, analysis of spectral kinetic energy fluxes in satellite altimeter data provides strong evidence of the occurrence of an inverse energy cascade in the ocean [3]. From the point of view of statistical mechanics, 2d turbulence is a prototype of non-equilibrium systems whose steady state is not described by Boltzmann statistics.

At variance with its three-dimensional counterpart for which the total kinetic energy is the unique inviscid invariant, the two-dimensional Navier–Stokes equation also preserves the total enstrophy (in the absence of forcing and dissipation) [29]. Enstrophy conservation is a key ingredient for the proof [41] of the existence and uniqueness of the solution of the Cauchy problem for the 2d Navier–Stokes equation with deterministic forcing. Very recently, this result has been extended to stochastic stirring. In particular, it was shown [39, 15, 24, 16, 17, 50] that the solution is a Markov process exponentially mixing in time and ergodic with a unique invariant (steady state) measure even when the forcing acts only on two Fourier modes [31].

A *phenomenological* theory proposed by Kraichnan [36] and further extended by Batchelor [4] and Leith [43] (see also [38, 5, 6, 44, 30]) predicts the presence of a double-cascade mechanism governing the transfer of energy and enstrophy in the limit of infinite inertial range. Accordingly, an *inverse energy cascade* with spectrum characterized by a scaling exponent $-5/3$ appears for values of the wavenumber p smaller than p_f, the typical forcing wavenumber. For wavenumbers larger than p_f a *direct enstrophy cascade* should occur. In this regime the energy spectrum should have a power-law exponent equal to -3 plus possible logarithmic corrections hypothesized in [37] to ensure constancy of the enstrophy transfer rate. Very strong laboratory experiments reviewed in [65, 35] and numerical experiments (see, e.g., [12] and references therein) corroborate Kraichnan's theory. A long-standing hypothesis [58], which has recently found support through numerical experiments [8, 9], also surmises the existence of a conformal invariance underlying the inverse energy cascade of 2d turbulence.

Despite these successes, a *first-principles* derivation of the statistical properties of 2d turbulence is still missing. An attempt in this direction has recently been undertaken [33, 32, 34, 2] by inquiring about the scaling properties of the velocity field and of the transported scalar field (passive scalar) when they are sustained by a random Gaussian forcing with self-similar spatial statistics. The Hölder exponent ε of the forcing correlation provides an order parameter interpolating between small scale thermal stirring and large scale stirring.

In three (and more) dimensions ultraviolet renormalization group analysis [26, 22] of this model yields the result $1-4\varepsilon/3$ for the scaling exponent of the kinetic energy spectrum holding to *all orders* in a perturbative expansion in powers of ε. Kolmogorov scaling [29] is recovered when the energy input becomes dominated by its infrared components at ε equal to two. The results of [26, 22] are coherent with physical intuition because only the case $\varepsilon = 2$ is a model of fully developed turbulence. Recent numerical simulations validate within the available resolution such a picture [64, 10].

The extension of renormalization group analysis to the 2d case is instead not straightforward and was only achieved in [33]. Although the prediction for the kinetic energy scaling exponent is the same as in the three-dimensional case, the result cannot be easily reconciled with the phenomenological intuition based on Kraichnan's theory. The latter suggests the onset of an inverse energy cascade already at $\varepsilon = 0$ when the energy and enstrophy input are dominated by the ultraviolet degrees of freedom.

The purpose of the present work is to shed light on the apparent contradiction between the phenomenological and the renormalization group theory. In doing so, we extend and complete results presented in a previous letter [51]. In particular, in section 2 we illustrate the details and interpretation of the model. In section 3 we summarize the results of the renormalization group analysis found in [33, 32, 34, 2]. In section 4 we draw on [5, 6] to solve the Kármán–Howarth–Monin equation under the hypotheses of Kraichnan's theory. In doing so, we focus our attention first on the parametric range where a direct comparison with the renormalization group theory is possible. We also briefly discuss the predictions of the phenomenological theory for the general case, as a benchmark for our numerical experiments. In section 5 we compare the renormalization group predictions with the outcomes of a direct numerical integration of the model equations. There we give clear evidence that in all parametric ranges only the phenomenological theory *à la* Kraichnan is able to describe the observed behavior of the stochastic flow. Finally, in section 6 we discuss possible mechanisms underlying the discrepancy between the renormalization group predictions and the observed scaling behavior of the model.

2. The model

We consider the 2d Navier–Stokes equation governing the evolution of the velocity $\boldsymbol{v}$ of an incompressible Newtonian fluid:

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\partial_x})\boldsymbol{v} = \nu\partial_{\boldsymbol{x}}^2\boldsymbol{v} - \boldsymbol{\partial_x}P - \frac{\boldsymbol{v}}{\tau} + \boldsymbol{f} \qquad (1)$$

$$\boldsymbol{\partial_x}\cdot\boldsymbol{v} = \boldsymbol{\partial_x}\cdot\boldsymbol{f} = 0 \qquad (2)$$

and the forced advection–diffusion equation for a scalar (concentration) field θ:

$$(\partial_t + \boldsymbol{v}\cdot\boldsymbol{\partial_x})\theta = \kappa\partial_x^2\theta + g \qquad (3)$$

where ν is the kinematic viscosity and κ is the diffusivity of the scalar field. The Ekman friction term $-\boldsymbol{v}/\tau$ included in equation (1) ensures that a steady state is attained by damping kinetic energy transfer towards larger and larger scales [25]. Both equations (1) and (3) are sustained by stochastic forcing fields, respectively $\boldsymbol{f}$ and g, with Gaussian statistics such that

$$\prec \boldsymbol{f}(\boldsymbol{x},t) \succ = \prec g(\boldsymbol{x},t) \succ = 0 \qquad (4)$$

and

$$\prec f^\alpha(\boldsymbol{x},t)f^\beta(\boldsymbol{y},s) \succ = \delta(t-s)F^{\alpha\beta}(\boldsymbol{x}-\boldsymbol{y},m_f,M_f), \qquad \alpha,\beta = 1,2 \qquad (5)$$

$$\prec g(\boldsymbol{x},t)g(\boldsymbol{y},s) \succ = \delta(t-s)G(\boldsymbol{x}-\boldsymbol{y},m_g,M_g). \qquad (6)$$

Whilst time decorrelation in (5) and (6) is meant to preserve Galilean invariance of the statistics (in the absence of Ekman friction), the spatial part of the forcing correlations is chosen to be isotropic and self-similar in a wavenumber range between well-separated infrared m_f, m_g and ultraviolet M_f, M_g cutoffs. Specifically

$$F^{\alpha\beta}(\boldsymbol{x},m_f,M_f) = F_o\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{p^{2\varepsilon-2}}\Pi^{\alpha\beta}(\hat{\boldsymbol{p}})\chi_f\left(\frac{m_f^2}{p^2},\frac{p^2}{M_f^2}\right), \qquad \frac{m_f^2}{M_f^2}\ll 1 \qquad (7)$$

with $\Pi^{\alpha\beta}$ the transversal projector in Fourier space, ε the Hölder exponent and

$$G(\boldsymbol{x},m_g,M_g) = G_o\int\frac{\mathrm{d}^2p}{(2\pi)^2}\frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{p^{2\,h-2}}\chi_g\left(\frac{m_g^2}{p^2},\frac{p^2}{M_g^2}\right), \qquad \frac{m_g}{M_g}\ll 1 \qquad (8)$$

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with Hölder exponent h. The explicit form of the isotropic cutoff functions χ_f and χ_g is unimportant as far as they remain approximately constant for any wavenumber $\boldsymbol{p}$ in the scaling range $m_f, m_g \ll p \ll M_f, M_g$. It is not restrictive to choose χ_f and χ_g normalized to unity in the origin. Note that at ε and h equal zero the traces of (7) and (8) are proportional to the Laplacian of a Dirac δ function centered at the origin. More generally, the Hölder exponents ε and h determine the spectral composition of the energy injection. For the velocity field one finds

$$I_{\mathcal{E}}(m_f, M_f) = F_o \int \frac{\mathrm{d}^2 p}{(2\pi)^2} p^{2-2\varepsilon} \chi_f\left(\frac{m_f^2}{p^2}, \frac{p^2}{M_f^2}\right) \propto \begin{cases} M_f^{4-2\,\varepsilon} I_{\mathcal{E}}(0,1), & 0 \leq \varepsilon < 2 \\ m_f^{4-2\varepsilon} I_{\mathcal{E}}(1,0), & \varepsilon > 2. \end{cases} \tag{9}$$

Similarly, the injection for the scalar field is dominated by wavenumbers around the ultraviolet (infrared) cutoff for any $h < 2$ ($h > 2$). For the purposes of the present analysis it is worth recalling the vorticity representation of the Navier–Stokes equation in 2d:

$$(\partial_t + \boldsymbol{v} \cdot \partial_{\boldsymbol{x}})\,\omega = \nu \partial_{\boldsymbol{x}}^2 \omega - \frac{\omega}{\tau} + f_\omega \tag{10}$$

where

$$\begin{aligned} &\omega = \epsilon_{\alpha\beta} \partial_{x_\alpha} v^\beta \qquad \text{and} \qquad f_\omega = \epsilon_{\alpha\beta} \partial_{x^\alpha} f^\beta \\ &\epsilon_{1\,2} = -\epsilon_{2\,1} = 1 \qquad \text{and} \qquad \epsilon_{1\,1} = \epsilon_{2\,2} = 0. \end{aligned} \tag{11}$$

Equation (10) implies the conservation of the total enstrophy $\mathcal{Z}$:

$$\mathcal{Z} = \int \mathrm{d}^2 x \prec \omega(\boldsymbol{x}, t) \omega(\boldsymbol{0}, t) \succ = -\int \mathrm{d}^2 x\, \partial_{\boldsymbol{x}}^2 \prec \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(\boldsymbol{0}, t) \succ \tag{12}$$

whenever the right-hand side of (10) is negligible. For power-law forcing, the enstrophy injection is also controlled by the Hölder exponent ε:

$$I_{\mathcal{Z}}(m_f, M_f) = F_o \int \frac{\mathrm{d}^2 p}{(2\,\pi)^2} p^{4-2\,\varepsilon} \chi_f\left(\frac{m_f^2}{p^2}, \frac{p^2}{M_f^2}\right) \propto \begin{cases} M_f^{6-2\,\varepsilon} I_{\mathcal{Z}}(0,1), & 0 \leq \varepsilon < 3 \\ m_f^{6-2\varepsilon} I_{\mathcal{Z}}(1,0), & \varepsilon > 3. \end{cases} \tag{13}$$

The relations (9) and (13) show that the energy and enstrophy injections are simultaneously concentrated in the ultraviolet and in the infrared only for $\varepsilon < 2$ and $\varepsilon > 3$, respectively.

3. Summary of the renormalization group analysis results

Let d_A denote the scaling dimension of a physical quantity A. Crudely matching canonical dimensions in (1) and (3) suggests, irrespective of the spatial dimension, the existence of two scaling ranges: a dissipative one for wavenumbers such that nonlinear effects are negligible:

$$d_t = 2 d_x \qquad \text{and} \qquad d_v = -d_x(1-\varepsilon) \qquad \text{and} \qquad d_\theta = -d_x(1-h) \tag{14}$$

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and an inertial range corresponding to the requirement of Galilean invariance imposed by matching the two terms of the material derivative $D_t := \partial_t + \boldsymbol{v} \cdot \boldsymbol{\partial_x}$ with the forcing:

$$d_t = d_x \left(2 - \frac{2\varepsilon}{3}\right) \qquad \text{and} \qquad d_v = -d_x \left(1 - \frac{2\varepsilon}{3}\right) \qquad \text{and} \qquad d_\theta = -d_x \left(1 - h + \frac{\varepsilon}{2}\right). \tag{15}$$

For ε, h equal zero the scaling dimensions (14) and (15) coalesce. Physically, the coalescence point corresponds to stirring by thermal noise. Coalescence hints at the existence of a marginal case in the renormalization group sense. This means (see, e.g., [70]) that scaling dimensions at small but finite ε, h may be obtained in a Taylor series based on their values in the marginal case in analogy to equilibrium critical phenomena where marginality is defined by an upper critical dimension specified, for example, by Ginzburg's criterion. A similar scenario seems to apply to (1) for $d > 2$ and in the absence of large scale friction (τ set to infinity). In such a case [26, 22, 1], fine-tuning the amplitude of the forcing correlation (7) to be $O(\varepsilon)$ yields an expansion in powers of ε around a Gaussian theory specified by renormalized eddy diffusivities ν_{R} and κ_{R}. Ultraviolet renormalization guarantees that the eddy diffusivities are related to the molecular viscosities appearing in (1) and (3) by renormalization constants:

$$Z_\nu := \frac{\nu}{\nu_{\mathrm{R}}} \qquad \text{and} \qquad Z_\kappa := \frac{\kappa}{\kappa_{\mathrm{R}}} \tag{16}$$

determined at any order in perturbation theory by subtracting all 'resonant' terms diverging with the ultraviolet cutoffs M_f and M_g. Technically, this is achieved by identifying the ultraviolet-divergent part of the one-particle irreducible vertex associated with the response functions of the velocity and concentration fields [1]. The result is that, within all order accuracy in ε, all correlation functions of velocity and concentration fields sampled at well-separated spatial points scale according to canonical dimensional predictions. In particular, the scaling laws (14) and (15) correspond to an ultraviolet-and an infrared-stable fixed point of the renormalization group transformation respectively describing the dissipative and inertial ranges. Extending this analysis to the 2d case presents extra difficulties. Already in the absence of nonlinearities, the correlation function is simultaneously logarithmically divergent both in the infrared and in the ultraviolet. The Ekman term in (1) is then needed to decouple infrared degrees of freedom. Once this is done, dimensional analysis shows that the renormalization constants (16) are not sufficient to reabsorb all terms divergent with the ultraviolet cutoffs in the perturbative solution of (1). This is a serious difficulty because direct calculations [2, 69] hint that ultraviolet renormalization group transformations using *non-local* counter-terms may lead to mathematical inconsistencies. In other words, renormalization constants should only be associated with coupling constants of local interactions in real space. For (1) the molecular viscosity is the only coupling constant satisfying such a requirement [1, 33, 2]. This difficulty led to a controversy, summarized in [33], about the very possibility of applying renormalization group methods to 2d turbulence. In [33] it is also argued that multiplicative ultraviolet renormalization remains consistent to all orders in perturbation theory if the forcing correlation is modified to include a local (analytic) component:

$$F^{\alpha\beta}(\boldsymbol{x}, m_f, M_f) \to F^{\alpha\beta}(\boldsymbol{x}, m_f, M_f) + F^{\alpha\beta}_{(\mathrm{local})}(\boldsymbol{x}, m_f, M_f) \tag{17}$$

with

$$F^{\alpha\beta}_{(\mathrm{local})}(\boldsymbol{x}, m_f, M_f) := F^{(\mathrm{local})}_o \int \frac{\mathrm{d}^2 p}{(2\pi)^2} \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \Pi^{\alpha\beta}(\hat{\boldsymbol{p}}) p^2 \chi_f \left(\frac{m_f^2}{p^2}, \frac{p^2}{M_f^2} \right). \tag{18}$$

From the renormalization group point of view, the replacement is justified by the observation that the resulting model generates the same relevant couplings as the original and therefore should fall in the same universality class. The merit of (17) is to provide through $a_o := F^{(\mathrm{local})}_o/\nu^3$ the zeroth order of the extra renormalization constant:

$$Z_a := \frac{a_o}{a} \tag{19}$$

needed to reabsorb all the remaining explicit dependence on M_f in the perturbative expansion of correlation functions of the velocity field. As a consequence, [33] predicts that the isotropic energy spectrum of the velocity field

$$E_{[\boldsymbol{v}]}(p) := \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \delta(q-p) \int \mathrm{d}^2 x \, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \prec \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{v}(0, t) \succ \tag{20}$$

admits the expression

$$E_{[\boldsymbol{v}]}(p) = \varepsilon^{1/3} \left(\frac{F_o}{\nu^3} \right)^{2/3} \nu^2 p^{1-4\varepsilon/3} R \left(\varepsilon, \frac{m_f}{p}, \left(\frac{p_b}{p} \right)^{2-2\varepsilon/3} \right). \tag{21}$$

In (21) the wavenumber

$$p_b \propto \left(\frac{\varepsilon}{\nu^3 \tau^3} \right)^{1/(6-2\varepsilon)} \tag{22}$$

signals whether at small scales dissipation is mainly due to friction ($p \ll p_b$) or to molecular viscosity ($p \gg p_b$). The function R has a regular expansion in ε for fixed p_b. The infrared scaling of (21) is then determined by the behavior of R in the limit $p \downarrow 0$. This limit is inquired within the renormalization group formalism by the so-called operator product expansion. The outcome [33] is that the energy spectrum admits the same infrared asymptotics as in dimensions higher than two:

$$E_{[\boldsymbol{v}]}(p) \sim p^{1-4\varepsilon/3}. \tag{23}$$

It is worth emphasizing that the above results were derived in [33] using the vorticity representation of the velocity field which holds only in 2d. As a further check, in [34, 32] the same results were recovered by analytic continuation of (1) in the limit $d \downarrow 2$. In [32] the analysis extends to the scaling properties of the concentration field and gives

$$E_{[\theta]}(p) := \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \delta(p-q) \int \mathrm{d}^2 x \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \prec \theta(\boldsymbol{x}, t)\theta(0, t) \succ \sim p^{1+(2\varepsilon/3)-2h} \tag{24}$$

in agreement with the dimensional prediction (15). In summary, according to the renormalization group analysis of [33, 34, 32], in 2d as in 3d the scaling properties of (1) and (3) differ for ε tending to zero from those of fully developed turbulence. In particular, inverse cascade-like scaling is attained only for $\varepsilon = 2$ and direct cascade-like at $\varepsilon = 3$. In the following section we will argue that these results are in contradiction with those that a phenomenological theory *à la* Kraichnan would suggest.

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4. Phenomenology *à la* Kraichnan

Our starting point is the Kármán–Howarth–Monin equation [29]:

$$\tfrac{1}{2}\partial_\mu \prec \delta v^\mu(\boldsymbol{x},t)||\delta \boldsymbol{v}(\boldsymbol{x},t)||^2 \succ \\ = \left(\partial_t + \frac{2}{\tau}\right) \prec v^\alpha(\boldsymbol{x},t) v_\alpha(\boldsymbol{0},t) \succ + 2\nu \prec \partial_\mu v^\alpha(\boldsymbol{x},t)\, \partial^\mu v_\alpha(\boldsymbol{0},t) \succ - F^\alpha_\alpha(\boldsymbol{x}) \tag{25}$$

and the analogous expression for the scalar field:

$$\tfrac{1}{2}\partial_\mu \prec \delta v^\mu(\boldsymbol{x},t)\, [\delta\theta(\boldsymbol{x},t)]^2 \succ \\ = \partial_t \prec [\delta\theta(\boldsymbol{x},t)]^2 \succ + 2\kappa \prec \partial_\mu \theta(\boldsymbol{x},t)\, \partial^\mu \theta(\boldsymbol{0},t) \succ - G(\boldsymbol{x}). \tag{26}$$

In (25) and (26) the notation is

$$\delta v^\mu(\boldsymbol{x},t) := v^\mu(\boldsymbol{x},t) - v^\mu(\boldsymbol{0},t) \qquad \delta\theta(\boldsymbol{x},t) := \delta\theta(\boldsymbol{x},t) - \delta\theta(\boldsymbol{0},t). \tag{27}$$

The scaling predictions of Kraichnan's theory stem from the asymptotic solution of (25) under the following three assumptions [7]:

(i) velocity correlations are smooth at finite viscosity and exist in the inviscid limit even at coinciding points,
(ii) even in the absence of large scale friction (i.e. $\tau = \infty$) Galilean invariant functions, and in particular structure functions, reach a steady state,
(iii) no dissipative anomalies occur for the energy cascade.

The assumptions (i) and (ii) imply that the two-point correlation in the absence of Ekman friction does not reach a steady state:

$$\prec v^\alpha(\boldsymbol{x},t) v_\alpha(\boldsymbol{0},t) \succ = \lambda t - \tfrac{1}{2} \prec ||\delta \boldsymbol{v}(\boldsymbol{x},t)||^2 \succ + \cdots (\tau = \infty) \tag{28}$$

the constant λ being the asymptotic growth rate. By (iii) energy dissipation in (25) satisfies

$$\{\lim_{\nu\downarrow 0} \lim_{x\downarrow 0} - \lim_{x\downarrow 0} \lim_{\nu\downarrow 0}\} \nu \prec \partial_\mu v^\alpha(\boldsymbol{x},t) \partial^\mu v_\alpha(\boldsymbol{0},t) \succ = 0. \tag{29}$$

This latter hypothesis is distinctive of two-dimensional turbulence: in three and higher dimensions the limits are not expected to commute for fully developed turbulence. If the bulk of the energy injection $I_\mathcal{E}$ occurs around a wavenumber p_f and viscosity and friction are such that the adimensional parameter

$$\mathcal{R} = \frac{I_\mathcal{E} \tau^2}{\nu} \gg 1 \tag{30}$$

plays the role of a large Reynolds number then the three hypotheses yield an inverse cascade for wavenumbers $\boldsymbol{p}$ in the range $p_\tau \ll p \ll p_f$ with $p_\tau = (I_\mathcal{E}\tau^3)^{-1/2}$ and a direct cascade for $p_f \ll p \ll \bar{p}_\tau$ with $\bar{p}_\tau = (\nu\tau)^{-1/2} = p_\tau/\mathcal{R}^{1/2}$ [5, 6]. Note that the Kolmogorov scale $p_\mathrm{K} =: (I_\mathcal{E}/\nu^3)^{1/4} = p_\tau/\mathcal{R}^{3/4}$ is always smaller than the dissipation scale set by the Ekman friction. We show below how the same arguments can be adapted to a power-law forcing.

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4.1. Inverse cascade: $\varepsilon < 2$

By (9) and (13) the energy and the enstrophy input are in this case dominated by the ultraviolet cutoff M_f. Neglecting m_f, the trace of the forcing correlation admits the asymptotic expansion (see appendix A for details)

$$F_\alpha^\alpha(\boldsymbol{x},0,M_f) = \begin{cases} M_f^{4-2\varepsilon}\left\{ I_{\mathcal{E}}(0,1) - \dfrac{I_{\mathcal{Z}}(0,1)(M_f x)^2}{2} + \cdots \right\} & M_f x \ll 1 \\ \dfrac{4^{1-\varepsilon}\Gamma\,(2-\varepsilon)}{\pi\Gamma(\varepsilon-1)}\dfrac{F_o}{x^{4-2\varepsilon}} & M_f x \gg 1 \end{cases}$$

holding for $0 < \varepsilon < 2$. Comparison with the renormalization group results is possible in the infrared region, $M_f x \gg 1$. In order to extricate the corresponding asymptotics of the third-order structure function, it is convenient to consider first the quasi-stationary case for τ tending to infinity. By hypotheses (i) and (iii) [5], the asymptotic growth rate of (28) in the inviscid limit is equal to the energy injection

$$\lambda = M_f^{4-2\varepsilon} I_{\mathcal{E}}(0,1) := M_f^{4-2\varepsilon} F_o^\star. \tag{31}$$

By (25) the growth rate sustains the structure function at scales $M_f x \gg 1$:

$$\prec \delta v^\mu(\boldsymbol{x},t)||\delta v||^2(\boldsymbol{x},t) \succ = F_o^\star M_f^{4-2\varepsilon} x^\mu \left\{ 1 - \frac{4^{1-\varepsilon}\Gamma\,(2-\varepsilon)\,F_o}{\pi\Gamma(\varepsilon)F_o^\star(M_f x)^{4-2\varepsilon}} + \cdots \right\}. \tag{32}$$

In the presence of the Ekman friction ($\tau < \infty$) (25) reaches a steady state. In such a case the energy injection is balanced by the velocity correlation which, far from the infrared cutoff $p_\tau = (F_o M_f^{4-2\varepsilon}\tau^3)^{-1/2}$, is expected to take the form

$$\prec \boldsymbol{v}^\alpha(\boldsymbol{x},t)\boldsymbol{v}_\alpha(\boldsymbol{0},t) \succ = \frac{\tau F_0^\star M_f^{4-2\varepsilon}}{2}\{1 - c_1(p_\tau x)^{\zeta_2} + \cdots\} \tag{33}$$

with c_1 a pure number and ζ_2 to be determined by a self-consistency condition. The asymptotics of the structure function acquires a correction

$$\begin{aligned} &\prec \delta v^\mu(\boldsymbol{x},t)||\delta v||^2(\boldsymbol{x},t) \succ \\ &\quad = F_o^\star M_f^{4-2\varepsilon} x^\mu \left\{ 1 - \frac{2c_1(p_\tau x)^{\zeta_2}}{(2+\zeta_2)} - \frac{4^{1-\varepsilon}\Gamma\,(2-\varepsilon)\,F_o}{\pi\Gamma(\varepsilon)F_o^\star(M_f x)^{4-2\varepsilon}} + \cdots \right\}. \end{aligned} \tag{34}$$

Some remarks are in order. The constant flux solution dominates for

$$\frac{1}{M_f} \ll x \ll (F_o M^{4-2\varepsilon}\tau^3)^{1/2}. \tag{35}$$

In this range the renormalization group prediction clearly appears as a *sub-leading* correction. Similarly, the two-point correlation adds a further sub-leading term associated with the exponent ζ_2. Dimensional considerations yield for ζ_2 the value $2/3$ whence a $-5/3$ exponent follows for the energy spectrum. The sign of the constant flux term stemming from (32) and (34) is positive so describing energy transfer to larger scales. The conclusion is of an inverse cascade taking place above the forcing ultraviolet cutoff. In such a case, the study of the statistics of the passive scalar should recover the results of [11]. *A priori* inspection of (26) allows one to distinguish at least two sub-cases.

4.1.1. Scalar field in the inverse cascade and small scale forcing $h < 2$. For $h < 2$ the injection of scalar fluctuations is concentrated in the ultraviolet (thermal stirring). No dissipative anomaly is expected. The Kármán–Howarth–Monin equation (26) yields

$$\prec \delta v^{\mu}(\boldsymbol{x},t)\,[\delta\theta(\boldsymbol{x},t)]^2 \succ \simeq -\frac{4^{1-h}\Gamma(2-h)\,G_o x^{\mu}}{\pi\Gamma(h)x^{4-2h}} \tag{36}$$

in the scaling range

$$\tilde{m} = \left\{m_f^{-1}, m_g^{-1}\right\} \gg x \gg \min\left\{M_f^{-1}, M_g^{-1}\right\} := \tilde{M}. \tag{37}$$

Inferences about the correlation functions of the scalar field can be barely drawn from crude dimensional considerations. Accordingly, one has

$$E_{[\theta]}(p) \sim \frac{\prec [\delta\theta(\boldsymbol{p}/p^2,t)]^2 \succ}{p^{1-1/3}} \sim p^{(7-6h)/3} \tag{38}$$

which differs from the one stemming from the scaling dimension d_θ given in (15) and supported by the renormalization group calculations of [32]. In particular, whilst (15) recovers equipartition scaling only for $(\varepsilon, h) = (0, 0)$, (38) yields a scaling linear in wavenumber space at $h = 2/3$. This may indicate the breakdown for $h < 2/3$ of (38) and the onset of an equipartition-type scaling for the spectrum of the scalar field, see appendix B for quantitative modeling of the phenomenon.

4.1.2. Scalar field in the inverse cascade and large scale forcing $h > 2$. In this regime, the injection of the scalar field is dominated by the infrared cutoff. Physically the situation may be assimilated to turbulent stirring. The Kármán–Howarth–Monin equation for $m_g x \ll 1$ reduces to

$$\tfrac{1}{2}\partial_\mu \prec v^{\mu}(\boldsymbol{x},t)\,[\delta\theta(\boldsymbol{x},t)]^2 \succ \simeq -G_o m_g^{4-2h}\{\bar{\gamma}_{h-2} + (m_g x)^{2h-4}\bar{\gamma}_0 + (m_g x)^2\bar{\gamma}_{h-1} + \cdots\}. \tag{39}$$

The coefficients of the forcing expansion are specified by the formulae of appendix A by identifying γ with the function ϕ thereby defined. Equation (39) points at a direct cascade of the scalar field with sub-leading corrections due to the power-law forcing. The dimensional prediction for the spectrum of the scalar field is Obukhov–Corrsin's [29]

$$E_{[\theta]}(p) \sim p^{-5/3}. \tag{40}$$

The numerical experiments of [11] support Obukhov–Corrsin's scaling in this regime.

4.2. Local balance: $2 < \varepsilon < 3$

For $2 < \varepsilon < 3$ the injection of the scalar field is dominated by the infrared cutoff m_f. The quasi-steady-state solution for vanishing Ekman friction yields

$$\lambda = m^{4-2\varepsilon} I_{\varepsilon}(1,0) := F_o m_f^{4-2\varepsilon}\phi_{\varepsilon-2}. \tag{41}$$

Correspondingly, the scaling range is set by the condition $m_f x \ll 1$ alone and (25) becomes

$$\tfrac{1}{2}\partial_\mu \prec \delta v^{\mu}(\boldsymbol{x},t)||\delta\boldsymbol{v}(\boldsymbol{x},t)||^2 \succ = \lambda - F_o m_f^{4-2\varepsilon}\{\phi_{\varepsilon-2} + (m_f x)^{2\varepsilon-4}\phi_0 + (m_f x)^2\phi_{\varepsilon-1} + \cdots\}. \tag{42}$$

As the first two terms on the right-hand side of (42) cancel out, the third-order structure function admits the asymptotic expression

$$\prec \delta v^{\mu}(\boldsymbol{x},t)||\delta \boldsymbol{v}(\boldsymbol{x},t)||^2 \succ \simeq -F_o x^{\mu} x^{2\varepsilon-4}\left\{\frac{\phi_0}{\varepsilon-1}+\frac{(m_f x)^{6-2\varepsilon}\phi_{\varepsilon-1}}{2}+\cdots\right\} \tag{43}$$

with $\phi_0 < 0$ (see appendix A). The steady state solution in the presence of Ekman friction can be discussed as in section 4.1 and introduces in such a case only sub-leading terms. The cancellation in (42) was argued in [5]–[7] to underlie the direct cascade for standard turbulent forcing. The difference here is that power-law forcing dominates the scaling. In this range, the solution of (25) validates the dimensional prediction (15). The agreement extends to the scalar field with two provisos:

(i) $h < 2$: the threshold for equipartition scaling is $h^{\star} = \varepsilon/3 > 2/3$:

(ii) $h > 2$: the forcing becomes dominated by the infrared cutoff. Correspondingly, a '*freezing*' of the scaling dimensions at the value for $h = 2$ may be expected [28, 1].

In summary, the expected spectra are

$$E_{[\boldsymbol{v}]}(p) \sim p^{1-4\varepsilon/3} \qquad \text{and} \qquad E_{[\theta]}(p) \sim \begin{cases} p, & 0 < h < \frac{\varepsilon}{3} \\ p^{1+(2\varepsilon/3)-2h}, & \frac{\varepsilon}{3} < h < 2 \\ p^{-3+2\varepsilon/3}, & h > 2. \end{cases} \tag{44}$$

It should be noted that in this regime for $\varepsilon/3 < h < 2$ the predictions of the renormalization group and of the phenomenological theory coincide.

4.3. Direct cascade: $\varepsilon > 3$

For $\varepsilon > 3$ both energy (9) and enstrophy (13) injection are dominated by the infrared cutoff. The analysis of the Kármán–Howarth–Monin equation under (i), (ii) and (iii) in the quasi-steady-state follows the same lines as in previous section 4.2. The second term in the square brackets of (42) now dominates scaling:

$$\prec \delta v^{\mu}(\boldsymbol{x},t)||\delta \boldsymbol{v}(\boldsymbol{x},t)||^2 \succ \simeq -F_o x^{\mu} x^{2} m_f^{6-2\varepsilon}\left\{\frac{\phi_{\varepsilon-1}}{2}+\frac{(m_f x)^{2\varepsilon-6}\phi_0}{\varepsilon-1}+\cdots\right\}. \tag{45}$$

However, (45) is just the simplest of the possible scenarios. A detailed analysis of the direct cascade stirred by turbulent forcing [6] indicates that scaling in the steady state brought about by an Ekman friction may well be characterized by *non-universal* exponents, depending upon the value of τ. Subsequent numerical investigations [57, 13, 66] validate non-universal exponents in the steady state. A further inference drawn in [6] is the presence of logarithmic corrections to the quasi-steady-state structure function. For the scope of the present work it is sufficient to observe that the phenomenological theory predicts for $\varepsilon > 3$ the 'freezing' of the scaling dimension d_v to a value close to the 'naive' direct cascade ($d_v = d_x$). The implication for the energy spectra by dimensional arguments is

$$E_{[\boldsymbol{v}]}(p) \sim p^{-3+\cdots} \qquad \text{and} \qquad E_{[\theta]}(p) \sim \begin{cases} p, & 0 < h < 1 \\ p^{3-2h+\cdots}, & 1 < h < 2 \\ p^{-1+\cdots}, & h \geq 2. \end{cases} \tag{46}$$

In (46) the '...' denote non-universal and/or intermittent correction, the presence whereof is suggested by numerical investigations of the direct cascade stirred by turbulent forcing [57, 13, 66].

Observation

In the above discussion we omitted to discuss the properties of the flow of (1) and (3) at scales $M_f x \ll 1$. The reason for doing so, in the spirit of renormalization group and consistently with the numerical experiments of the ensuing section, is that the dynamics is strongly suppressed by dissipation effects for scales below M_f.

5. Numerical experiments

In order to compare the renormalization group predictions of section 3 with those of the phenomenological theory *à la* Kraichnan expounded in section 4, we performed numerical simulations of the Navier–Stokes equation for the vorticity field (10) and the advection–diffusion equation for the scalar field (3) with a fully dealiased pseudo-spectral method [18] in a doubly periodic square domain of size $L = 2\pi$ at resolution $N^2 = 1024^2$. Dealiasing cutoff is set to $k_t = N/3$. Time evolution was computed by means of a second-order Runge–Kutta scheme, with implicit handling of the linear friction and viscous terms. As is customary (see, e.g., [57, 66]) we added to (1) a hyperviscous damping $(-1)^{p-1}\nu_{p-1}\partial^{2p}v$. This is equivalent to a Pauli–Villars regularization, the use whereof is well justified in renormalization group calculations (see, e.g., [70, 33]). The integration time has been carried out for 20 large eddy turn-over times after the velocity fields have reached the stationary state. The stochastic forcing is implemented in Fourier space by means of Gaussian, white-in-time noise as in [13] but with variance determined according to (7).

In figure 1 we show the energy spectrum $E_{[\boldsymbol{v}]}$ for $\varepsilon = 0$. The numerical spectrum exhibits good agreement with the phenomenological theory of section 4 with a scaling exponent $d_{\mathcal{E}} = -5/3 d_p$ within numerical accuracy.

Such an exponent is very far from the equipartition-like scaling $d_{\mathcal{E}} = d_p$ which is the starting point for the renormalization group analysis. The energy flux validates the interpretation of the $-5/3$ spectrum as brought about by an inverse cascade. The energy flux

$$\Pi_{\mathrm{E}}(p, m_f) = \int_{m_f}^{p} \frac{\mathrm{d}^2 q}{(2\pi)^2} \, \mathrm{Re} \int \mathrm{d}^2 x \, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \prec \check{v}^{\alpha}(-\boldsymbol{q}, t)(v^{\beta}\partial_{\beta}v_{\alpha})(\boldsymbol{x}, t) \succ \tag{47}$$

where $\check{v}^{\alpha}$ denotes the Fourier transform of v^{α}, is *negative* and constant in the scaling range (see the inset of figure 1), so signaling the presence of an inverse cascade.

Breakdown of the marginality assumption at $\varepsilon = 0$ is confirmed also by the concentration spectra $E_{[\theta]}$ shown in figure 2: at $h = 1$ a spectrum $E_{[\theta]} \sim p^{1/3}$ well fits the numerical results, which definitely rule out the $E_{[\theta]} \sim p^{-1}$ prediction of the renormalization group theory (24). Furthermore, and in agreement with section 4.1, the spectrum of the scalar field undergoes a transition at $h = 2$. Above that value the scaling exponent freezes to the value $-5/3$ corresponding to a direct cascade, in agreement with the results of [11]. Figure 2 illustrates the phenomenon for $h = 2.5$. It should be noted that, for $h = 0$ an

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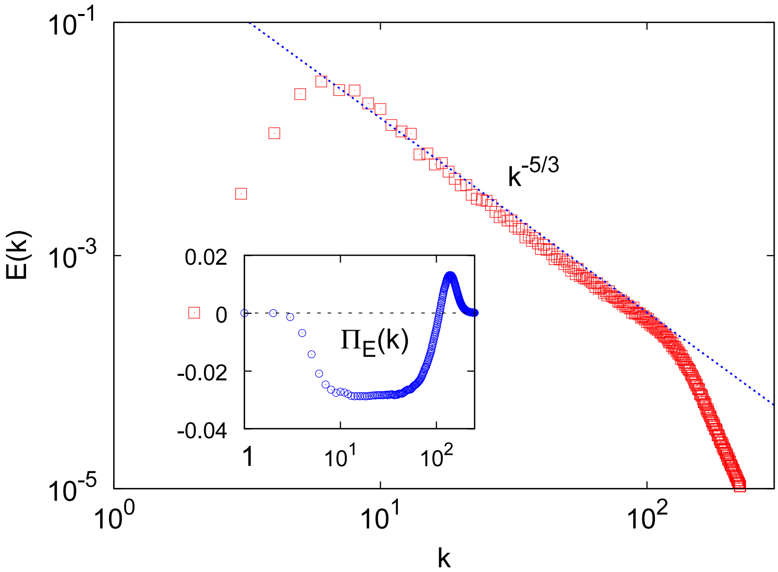

Figure 1. Kinetic energy spectrum for $\varepsilon = 0$. Inset: energy flux Π_{E}. Parameter values are: $m_f = 1, M_f = 341$, $\nu_3 = 10^{-16}$, $\tau_2^{-1} = 10^2$, $F_0 = 4 \times 10^{-10}$.

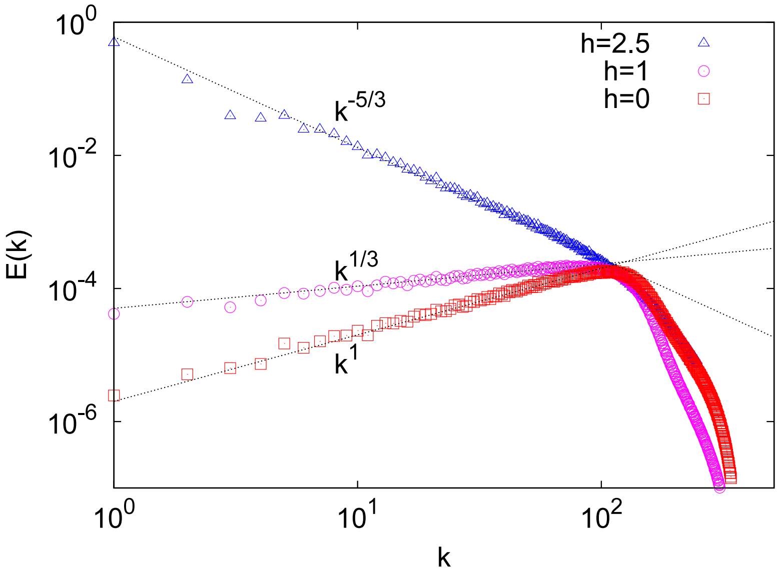

Figure 2. Scalar spectra for $\varepsilon = 0$ and various values of h. Parameters values are: $m_g = 1, M_g = 341$, $\kappa_3 = 10^{-16}$, $G_0 = 4 \times 10^{-10}$ for $h = 0$, $m_g = 1, M_g = 341$, $\kappa_3 = 10^{-16}$, $G_0 = 1.6 \times 10^{-5}$ for $h = 1$ and $m_g = 1, M_g = 341$, $\kappa_0 = 5 \times 10^{-4}$, $G_0 = 2.5 \times 10^{-1}$ for $h = 2.5$. Parameters for the velocity field as in figure 1.

equipartition-type scaling (i.e. linear in wavenumber) is observed for the spectrum of the scalar field. It is worth stressing that this is not sufficient to infer equipartition of the full statistics of the scalar field. It is known in analytically tractable cases of turbulent advection of scalar fields [27, 20, 21, 52] that equipartition-type scaling of the two-point correlation may well coexist with highly intermittent statistics even in the decay range of a scalar field.

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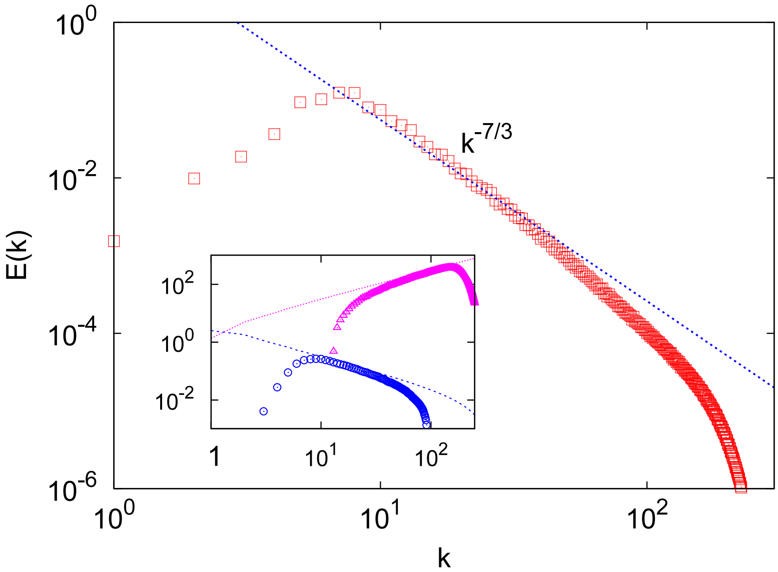

Figure 3. Kinetic energy spectrum $E(k)$ for $\varepsilon = 2.5$. Inset: energy flux Π_{E} (circles) multiplied by minus unity, and enstrophy flux Π_Z (triangles). The lines represents the injection spectra I_{E} (dashed line) and I_Z (dotted line). Parameters values are: $m_f = 1, M_f = 341$, $\nu_3 = 10^{-17}$, $\tau_2^{-1} = 10^3$, $F_0 = 1$.

In agreement with the phenomenological theory, for $2 < \varepsilon < 3$ no cascade is observed. Figure 3 illustrates the situation at $\varepsilon = 2.5$. The energy injection spectrum, defined as $I_{\mathrm{E}}(k) = \int_k^\infty \mathrm{d}p\, p^{3-2\varepsilon} \chi_f(m_f^2/p^2, p^2/M_f^2)$ is dominated by IR contributions, while the enstrophy injection spectrum $I_Z(k) = \int_0^k \mathrm{d}p\, p^{5-2\varepsilon} \chi_f(m_f^2/p^2, p^2/M_f^2)$ is still ultraviolet-divergent (see the inset of figure 3). In this situation the steady state is characterized by a scale-by-scale balance between the fluxes and the injection spectra. The resulting energy spectrum scales as $E_{[\boldsymbol{v}]}(p) \sim p^{1-4\varepsilon/3}$ (see figure 3). The spectra for the passive scalar shown in figure 4 are in agreement with the prediction (44).

Finally, figure 5 shows the onset of a direct cascade for $\varepsilon > 3$. The enstrophy flux:

$$\Pi_Z(k, m_f) = \int_{m_f}^{k} \frac{\mathrm{d}^2 q}{(2\pi)^2} \operatorname{Re} \int \mathrm{d}^2 x\, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{x}} \prec \check{\omega}(-\boldsymbol{q}, t)(v^\beta \partial_\beta \omega)(\boldsymbol{x}, t) \succ \tag{48}$$

is approximately constant and positive (indicating transfer towards smaller spatial scales) in the numerically resolved scaling range (see the inset of figure 5). It should be emphasized that numerical evidence from [57, 66, 13] uphold the non-universal dependence of the kinetic energy spectrum upon the Ekman friction in the direct cascade regime. Since the analysis of such effects lies beyond the scope of the present work we replaced for $\varepsilon > 2$ the Ekman friction with an hypo-dissipative term $(-1)^{q+1}\tau_q^{-1}\partial^{-2q}v$, which is expected to suppress the aforementioned non-universal corrections to the spectrum [46].

6. Discussions and conclusions

Our numerical experiments support, without possible ambiguity, the scenario set by Kraichnan's phenomenological theory. The physically relevant order parameters to

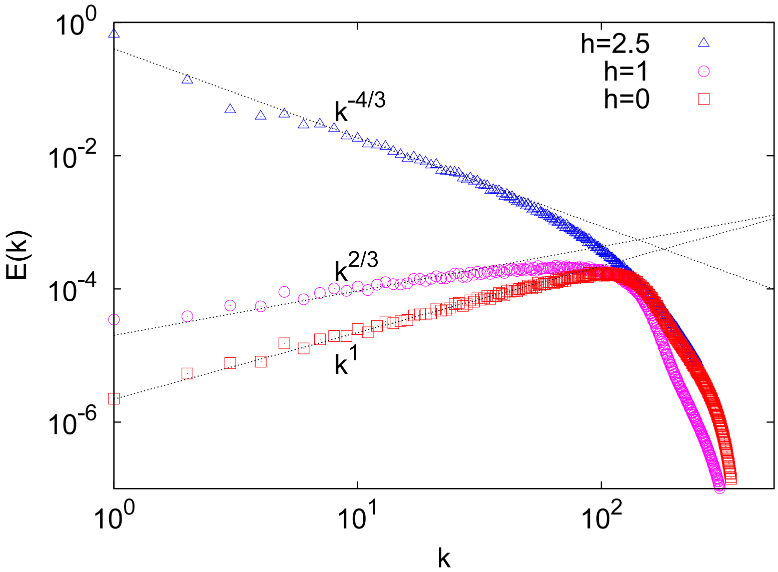

Figure 4. Scalar spectra for $\varepsilon = 0$ and various values of h. Parameter values are: $m_g = 1, M_g = 341$, $\kappa_3 = 10^{-16}$, $G_0 = 4 \times 10^{-10}$ for $h = 0$, $m_g = 1, M_g = 341$, $\kappa_3 = 10^{-16}$, $G_0 = 1.6 \times 10^{-5}$ for $h = 1$ and $m_g = 1, M_g = 341$, $\kappa_0 = 5 \times 10^{-4}$, $G_0 = 2.5 \times 10^{-1}$ for $h = 2.5$. Parameters for the velocity field as in figure 3.

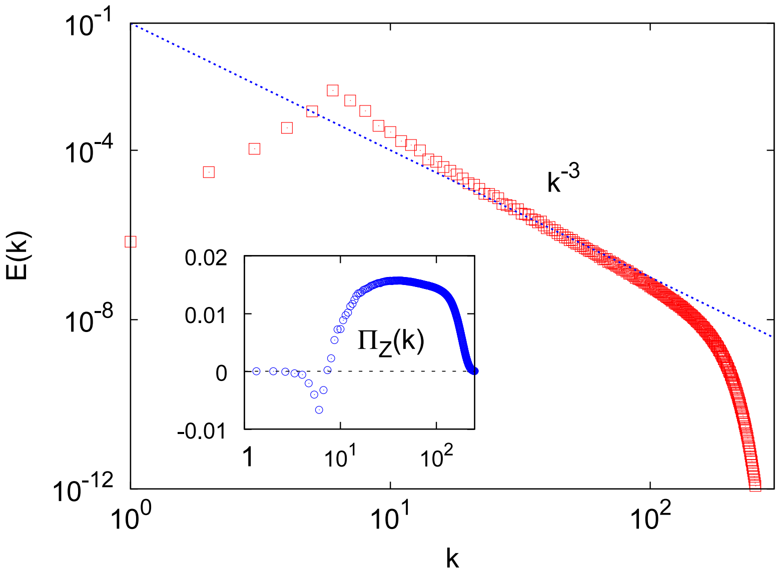

Figure 5. Kinetic energy spectrum for $\varepsilon = 4$. Inset: enstrophy flux Π_Z. Parameter values are: $m_f = 6, M_f = 240$, $\nu_3 = 10^{-18}$, $\tau_1^{-1} = 2$, $F_0 = 1$.

describe the qualitative behavior of (1) and (3) are the total energy (9) and enstrophy (13) injections. Whenever they coherently act on small (large) scales an inverse (direct) cascade is observed. In the intermediate case $2 < \varepsilon < 3$ no cascade takes place and a local balance scaling takes place. The renormalization group scaling exponents are for $\varepsilon, h < 2$ at most a sub-leading correction to the inverse cascade scaling. It remains to be clarified the origin of the discrepancy between the two theories. The renormalization group analysis of [33, 32, 34] is very thorough and satisfies all the self-consistency requirements that are known to produce correct scaling predictions in the theory of critical phenomena. Thus any trivial explanation of the discrepancy can be safely ruled out. The indication of the existence of a constant flux solution stems from the Kármán–Howarth–Monin equation. In the renormalization group language the relations closest to (25) are those satisfied by the composite operator algebra which includes the energy dissipation operator (see, e.g., section 2.2 of [1]). One may speculate that non-locality of forcing plays a role different than in 3d in determining the scaling dimensions of the elements of the operator algebra. It is, however, difficult to see how to consistently formalize this observation. One scenario that deserves to be further investigated is, in our opinion, the following. Perturbative renormalization implies the assumption that the infrared-stable fixed point governing the scaling regime emerges from a bifurcation at marginality from a Gaussian fixed point. Such an assumption is usually verified in critical phenomena but is not a necessary consequence of a general non-perturbative theory. In particular, there are examples of field theories where it is possible to give evidence that scaling is dominated by a fixed point emerging at marginality from bifurcations from non-perturbative, non-Gaussian fixed points. A concrete case is discussed in [49, 48][4]. There, a model of wetting transition indicates a scenario which could apply also to 2d turbulence. The scaling predictions associated with an infrared-stable fixed point captured by perturbative renormalization group analysis is numerically seen to be dominated by those associated with a second fixed point not bifurcating from the Gaussian fixed point at marginality. The existence of this fixed point can be exhibited only by a non-perturbative construction of the renormalization group transformation. The price to pay, however, is the introduction of truncations of the Wilson recursion scheme which cannot be *a priori* fully justified. If we accept such a point of view, the existence of the Kármán–Howarth–Monin equation should be interpreted as an *a priori* indication of the existence of a non-perturbative fixed point.

Acknowledgments

The authors are grateful to M Cencini, J Honkonen, A Kupiainen, L Peliti and M Vergassola for numerous discussions and insightful comments. This work was supported by the Center of Excellence *'Analysis and Dynamics'* of the Academy of Finland.

Appendix A. Asymptotics of the forcing correlation

Consider the scalar correlation

$$F(\boldsymbol{x}, m, M) = F_o \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{p^{d+\eta}} \chi\left(\frac{m^2}{p^2}, \frac{p^2}{M^2}\right) \tag{A.1}$$

with

$$\chi(0, 0) = 1. \tag{A.2}$$

[4] We thank L Peliti for drawing our attention to this point and to [49, 48].

In order to extricate the asymptotics in the range $m \ll p \ll M$ two cases should be distinguished depending upon the sign of η:

- If $\eta < 0$ the integral is infrared-and ultraviolet-convergent at finite point separations in the absence of cutoffs:

$$F(\boldsymbol{x}, m, M) \simeq F_o \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{p^{d+\eta}} = \frac{\Omega_d}{(2\pi)^d} \frac{x^\eta \Gamma(d/2)\Gamma(-\eta/2)}{2^{1+\eta}\Gamma((d+\eta)/2)} \tag{A.3}$$

  with Ω_d the solid angle in d dimensions.
- If $\eta > 0$ (A.1) is convergent only if the infrared cutoff is retained:

$$F(\boldsymbol{x}, m, M) \simeq F(\boldsymbol{x}, m, \infty) = F_o \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{p^{d+\eta}} \chi\left(\frac{m^2}{p^2}, 0\right). \tag{A.4}$$

  The integral can be estimated by inverting its Mellin representation with the help of the Cauchy theorem (see, e.g., [54]):

$$F(\boldsymbol{x}, m, \infty) = F_o \int_{\mathrm{Re}\,\zeta - \imath\infty}^{\mathrm{Re}\,\zeta + \imath\infty} \frac{\mathrm{d}\zeta}{(2\pi\imath)} x^\eta (mx)^{2\zeta} \phi(\zeta), \qquad \mathrm{Re}\,\zeta < -\frac{\eta}{2}. \tag{A.5}$$

  The holomorphic function

$$\phi(\zeta) := \frac{\Omega_d}{(2\pi)^d} \frac{\Gamma(d/2)\Gamma(-(2\zeta+\eta)/2)}{2^{1+2\zeta+\eta}\Gamma((d+2\zeta+\eta)/2)} \int_0^\infty \frac{\mathrm{d}w}{w} \frac{\chi(w^2, 0)}{w^\zeta} \tag{A.6}$$

  is by hypothesis analytic at least in a stripe for $\mathrm{Re}\,\zeta < -\eta/2$. Furthermore, by (A.2), the integral in (A.6) generates a simple pole for $\zeta = 0$. The Cauchy theorem yields for $mx \ll 1$ the asymptotics

$$F(\boldsymbol{x}, m, \infty) = F_o m^{-\eta} \left\{ \bar{\phi}_{-\eta/2} + (mx)^\eta \bar{\phi}_0 + (mx)^2 \bar{\phi}_{1-\eta/2} + \cdots \right\} \tag{A.7}$$

  having used the notation

$$\bar{\phi}_a := -\lim_{\zeta \uparrow a} (\zeta - a)\phi(\zeta). \tag{A.8}$$

Appendix B. Large scale zero-mode and power-law forcing in the Kraichnan model of advection of a concentration field

We refer the readers for definitions and details on the large scale decay properties of the Kraichnan model to [27, 20, 21, 52]. The energy spectrum of the Kraichnan model is exactly known. In 2d it takes, modulo irrelevant constant factors, for isotropic forcing the form

$$E_{[\theta]}(p) = \int_0^\infty \frac{\mathrm{d}\rho}{\rho} \frac{\rho B_1(p\rho)}{\kappa + D\rho^\xi} \int_0^\rho \frac{\mathrm{d}\sigma}{\sigma} \sigma^2 G(\sigma, m_g, M_g) \tag{B.1}$$

with $B_\eta(x)$ the Bessel function of order η, D the eddy diffusivity of the advecting velocity field and G given by (8). The Hölder exponent ξ is a free parameter in the model. Turbulent advection corresponds to $\xi = 4/3$. From (B.1) it is straightforward to check

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that

$$E_{[\theta]}(p) \sim \begin{cases} p\displaystyle\int_0^\infty \frac{\mathrm{d}\rho}{\rho}\frac{\rho^2}{\kappa + D\rho^\xi}\frac{\tilde{G}_o}{\rho^{2-2h}} = \left(\frac{\kappa}{D}\right)^{2h/\xi}\frac{\tilde{G}_o \pi p}{\kappa\xi\sin((2h\pi)/\xi)}, & h < \frac{\xi}{2} \\ p^{1+\xi-2h}\displaystyle\int_0^\infty \frac{\mathrm{d}\rho}{\rho}\frac{\tilde{G}_o B_1(\rho)}{D\rho^{1+\xi-2h}} = \frac{\tilde{G}_o p^{1+\xi-2h}\Gamma(h-\xi/2)}{2^{2+\xi-2h}D\Gamma(2-h+\xi/2)}, & h > \frac{\xi}{2}. \end{cases} \tag{B.2}$$

$\tilde{G}_o$ is a dimensional constant, the value of which is irrelevant for the present considerations. Setting $\xi = 4/3$ the results of section 4.1 of the main text are recovered. The direct cascade results (46) are recovered instead by setting $\xi = 2$. The asymptotics (B.2) hold under the assumption of infinite integral scale m_g^{-1} of the spatial forcing correlation. In the language of [20, 21] this means that the 'charge' $\check{G}(\mathbf{0}, 0, M_g)$ is *vanishing*. For $h > 2$ the forcing becomes infrared-dominated. At scales $m_g x \ll 1$

$$E_{[\theta]}(p) \sim G(0, m_g, \infty)p^{\xi-3}\int_0^1 \frac{\mathrm{d}\rho}{\rho}\rho^{3-\xi}\frac{B_1(\rho)}{D} \tag{B.3}$$

whilst in the opposite range $m_g x \gg 1$, for $\check{G}(\mathbf{0}, m_g, \infty) > 0$:

$$E_{[\theta]}(p) \sim \check{G}(0, m_g, \infty)p^{\xi-1}\int_0^\infty \frac{\mathrm{d}\rho}{\rho}\rho^{1-\xi}\frac{B_1(\rho)}{D}. \tag{B.4}$$

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