A STUDY OF THE REVERSIBLE PENDULUM.¹

PART I: THEORETICAL CONSIDERATIONS.

By John C. Shedd and James A. Birchby.

If the entire mass of a compound pendulum be $M$ and its radius of gyration about the centroid of the pendulum be $a$, its moment of inertia about the centroid is $Ma^2$ and its moment of inertia about a parallel axis at a distance $z$ is $M(a^2 + z^2)$. If this second axis be taken as the center of support of the pendulum, then its equivalent length is

$$l = z + \frac{a^2}{z}.$$  \hspace{1cm} (1)

Solving for $z$ the expression is had,

$$z = \frac{1}{2}(l \pm \sqrt{l^2 - 4a^2}).$$ \hspace{1cm} (2)

From this it appears that for any given equivalent length there are in general two possible positions of the supporting knife edge with reference to the centroid, i. e.,

$$z_1 = \frac{1}{2}(l + \sqrt{l^2 - 4a^2}),$$
$$z_2 = \frac{1}{2}(l - \sqrt{l^2 - 4a^2}).$$ \hspace{1cm} (3)

If the pendulum be provided with two knife-edges $k_1$ and $k_2$ then the equivalent lengths from the respective knife-edges are

$$L_1 = z_1 + \frac{a^2}{z_1},$$
$$L_2 = z_2 + \frac{a^2}{z_2}.$$ \hspace{1cm} (4)

These lengths will in general be different in value but it is of course possible that $l_1 = l_2$.

If this be the case, then $a^2 = \frac{s_1 s_2 (s_1 - s_2)}{(s_1 - s_2)} = s_1 s_2$, provided (in general) that $s_1 \neq s_2$. It is thus seen that the radius of gyration is the mean proportional between the segments $(s_1$ and $s_2$) into which the centroid divides the distance between the knife-edges.

Substituting the value $s_1 s_2$ for $a^2$ in the expression for $l_1$ and $l_2$, it is seen that

$$l_1 = l_2 = s_1 + s_2,$$

showing that the equivalent length now becomes the distance between knife-edges. Since $a$ is the mean proportional between $s_1$ and $s_2$, it is apparent that its value cannot exceed

$$\frac{1}{2} (s_1 + s_2), \quad \text{or} \quad \frac{1}{2} L.$$

If this condition with respect to $a$ be not fulfilled, the same period cannot be secured from both knife-edges, and the pendulum cannot be brought into proper adjustment.

The fact that it is possible to realize the condition $l = s_1 + s_2$ was made use of by Captain Kater in 1817, and constitutes the basis of the reversible pendulum bearing his name. Since then the general theory of the compound pendulum up to the point that Kater applied it has been completely worked out and may be found in various works on mechanics. This also includes a discussion of precautions and corrections necessary when the pendulum is used as an instrument of precision. In a recent article, Dr. R. R. Tatnall has somewhat extended the theoretical discussion to cover a number of important points. In the present paper it is proposed to discuss the equation of the pendulum from both the analytical and graphical standpoints, and also the various possible methods by which the pendulum may be used. Under the latter head several novel modifications will be pointed out.

From equation (1), together with the definition of $s$, it is seen that $l$ (and therefore the period of the pendulum), is independent of the direction in which $s$ is measured, provided only that it lie in a plane passing through the centroid of $M$ and normal to the axis

about which the moment of inertial is taken. Two circles may
thus be drawn about the centroid (Fig. 1, C) having radii equal to
\( s_1 \) and \( s_2 \) respectively, and all points on either of these circles taken
as points of suspension will give the same equivalent length, and hence the same pe-
riod for the pendulum. The values of the
radii of these circles are given by equa-
tions (3). In practice the points \( A, B, D \)
and \( E \) are the only points realizable and it
is seen that for every compound pendulum
there are four such points having the same
equivalent length and the same period.
The value of this common period is,

\[ t = \pi \sqrt{\frac{I}{g}} \]  
(5)

In general experimental work the point \( A \) is taken as fixed by one
knife-edge and the other knife-edge is moved until experimentally
located at \( B \). Under this condition \( l = s_1 + s_2 \) which equals the
distance between knife-edges. Or both knife-edges are fixed and
the movable masses of the pendulum are adjusted until the above
condition is realized. This is signalized by an equality in the period
taken from either knife-edge. Let us suppose, however, that in the
adjustment the knife-edges fall at \( B \) and \( D \) respectively. It is evi-
dent that the distance between knife-edges is no longer the equivalent
length of the pendulum, being now too short; similarly, if the
knife-edges fall at \( A \) and \( E \) the distance between knife-edges is too
great. Under these conditions the relation between the distance \( L \)
between knife-edges and the equivalent length \( l \) is given by the equation

\[ L = l \pm \sqrt{a^2 - 4a^2}. \]  
(6)

A special case arises when \( l = 2a \) and \( L = l \); under this condition
it is seen from (3) that \( s_1 = s_2 \) and that the centroid is located mid-
way between the knife-edges. The further condition is involved
that \( a = \frac{1}{2} L \). Graphically this condition is expressed in Fig. 1 by
the coalescence of the two circles. It is thus seen that it is pos-
sible (contrary to the general statements found in text-books) to have the centroid midway between the knife-edges and yet to bring the pendulum into adjustment.

The Equation of the Pendulum.—Let Fig. 2 represent a reversible pendulum having fixed knife-edges $k_1$ and $k_2$, and a movable mass $m$, all other parts being fixed.

Let

- $M$ = the mass of the pendulum aside from $m$,
- $b$ = the distance from $k_1$ to $C_v$, the centroid of $M$,
- $z_1$ = the distance from $k_1$ to $C_v$, the centroid of $M + m$,
- $z_2$ = the distance from $k_2$ to $C_v$, the centroid of $M + m$,
- $d$ = the distance from $k_1$ to $C_v$, the centroid of $m$,
- $I_x = Ma^2$ = the moment of inertia of $M$ about $C_v$,
- $I = (M + m)a^2$ = the moment of inertia of $M + m$ about $C_v$,
- $I_m = m\beta^2$ = the moment of inertia of $m$ about $C_v$,
- $a$ = the radius of gyration of $M + m$ about $C_v$,
- $R = m/M$,

Then

$$a^2 = I/(M + m)$$; whence from equation (1),

$$l_1 = z_1 + I/z_1(M + m)$$. But since $I = I_x + I_m + M(z_1 - b)^2 + m(d - z_1)^2$,

therefore

$$l_1 = z_1 + I_x + I_m + M(z_1 - b)^2 + m(d - z_1)^2 / z_1(M + m)$$; (7)

Whence

$$t = \pi \sqrt{\frac{g}{\pi g}} = \pi \left( \frac{z_1 + I_x + I_m + M(z_1 - b)^2 + m(d - z_1)^2}{g^2(M + m)} \right)^{1/2}.$$ (8)

From the principle of moments, $M(z_1 - b) = m(d - z_1)$, or

$$z_1 = (Mb + md)/(M + m) = \frac{b + R}{1 + R}.$$ (9)

Squaring both sides of equation (8) and substituting from equation (9) it is seen that,

$$\frac{g_f^2}{\pi^2}(b + Rd) = Rd^2 + b^2 + a^2 + R\beta^2.$$ (10)
In equations (8) and (10) the pendulum is supposed to be suspended from \( k_i \). If now the pendulum be suspended from \( k_2 \) the terms \( b, z, \) and \( d \) appear as \( L - b, L - z, \) and \( L - d \), so that equation (10) becomes,

\[
\frac{g T_2}{\pi^3} [L(1 + R) - (b + Rd)] = R(L - d)^3 + (L - b)^3 + a^3 + R^2 \beta^2. \tag{11}
\]

Equations (10) and (11) may be called the equations of the reversible pendulum.

A discussion of these equations constitutes the problem of the reversible pendulum. In this discussion there is a choice of terms that may be taken as variables. The pairs of variables that will be investigated are (1) \( t \) and \( d \), (2) \( t \) and \( \beta \), (3) \( t \) and \( R \). Inspection shows that \( b \) is not independent of \( d \), and that a change in \( a \) has essentially the same effect as a change in \( \beta \), so that the above pairs are sufficient to cover the problem.

Division I. \( t \) and \( d \) Taken as Variables.

If in equations (10) and (11) the origin be moved to

\[
\left(-\frac{b}{R}, \ 0\right) \text{ and } \left(\frac{L}{R} - \frac{b}{R} + L, \ 0\right)
\]

respectively, and in (11) \( d \) be changed to \(-d\), the equations reduce respectively to

\[
\begin{align*}
t_i^2d &= A d^2 + B d + C \\
t_a^2d &= A' d^2 + B' d + C'
\end{align*}
\tag{12}
\]

where the constants have the following values:

\[
A = \frac{\pi^2}{g} ; \quad B = -\frac{2\pi b}{R} g ; \quad C = \frac{\pi^3}{g} \left[\frac{(1 + R)b^2}{R^3} + \frac{a^3}{R} + \beta^2\right].
\]

\[
A' = \frac{\pi^2}{g} ; \quad B' = \frac{2\pi^2 (L - b)}{R} g ; \quad C' = \frac{\pi^2}{g} \left[\frac{(1 + R)(L - b)^2}{R^3} + \frac{a^2}{R} + \beta\right]
\]

Applying the criterion suggested by Sir Isaac Newton for cubic curves of the form of equations (12) it is seen that the roots of both \( Ad^2 + Bd + C = 0 \), and \( A' d^3 + B' d + C' = 0 \) are imaginary, since
$B^2 - 4AC = \frac{4\pi^4}{g^2} \left[ \frac{b^2 + a^2}{R} + \beta^2 \right]$ 

and 

$B'^2 - 4A'C' = \frac{4\pi^4}{g^2} \left[ \frac{(L - b)^2 + a^2}{R} + \beta^2 \right],$

both of which quantities are necessarily negative. Therefore both (10) and (11) are of the fifty-third species in Newton’s classification. The form of this curve is represented in Fig. 3. Since the right-hand side of (10) is always positive, $t_i$ is real for any value of $d$ which makes the left side positive. The condition for real values of $t_i$ then becomes:

$$d > -\frac{b}{R}. \quad (13)$$

It follows that if $b$ is positive $t_i$ is real for all positive values of $d$ and for negative values numerically smaller than $b/R$. If $b$ is negative $t_i$ is real only when $d$ is positive and numerically greater than $b/R$. For $d = -\frac{b}{R} t_i$ is infinite, and the curve has an asymptote parallel to the axis of $t$ at this point. (In Fig. 3 the origin has been shifted so that the $Y$ axis is this asymptote.) In the pendulum a negative value of $d$ means that the centroid of the movable mass is above $k_1$, and when it is at a distance $b/R$ above $k_1$ the centroid of the whole pendulum is at the supporting knife-edge. If
the movable mass be still further above \( k \), the pendulum is unstable, a condition indicated by an imaginary value for \( t_1 \) in equation (10).

In like manner \( t_2 \) in (11) is real for,

\[
 d < L + (L - b)/R 
\]  
(14)

It is easy to see that for the limiting value of \( d \) here given the centroid of the whole pendulum is at \( k \). The line \( d = L + (L - b)/R \) is an asymptote to the curve, the value of \( t_2 \) approaching infinity as \( d \) approaches this value.

Differentiating equation (10) and regarding \( t_1 \) as the dependent and \( d \) the independent variable the expression is obtained:

\[
 \frac{dt_1}{d(d)} = \frac{R(2\pi^2d - \beta t_1)}{2\beta t_1(Rd + \beta)} 
\]  
(15)

This is zero when

\[
 \frac{\beta t_1}{\pi^2} = 2d. 
\]

Eliminating \( t_1 \) between (10) and this equation the values of \( d \) for maxima or minima are had. These values are,

\[
 d = -\frac{b}{R} \pm \sqrt{\frac{\beta^2}{R^2} + \frac{\beta^2 + \alpha^2}{R} + \beta^2} 
\]  
(16)

The negative value of the radical gives an imaginary value for \( t_1 \) since it puts \( d \) beyond the limit assigned by (9). Hence there is but one such real point, and it may be shown that this corresponds to a minimum point for positive values, and a maximum for negative values of \( t_1 \). Since we are concerned only with conditions that may be experimentally realized, positive values of \( t_1 \) are alone considered.

In a similar manner, by differentiating equation (11) the expression is had:

\[
 \frac{dt_2}{d(d)} = \frac{R[\beta t_2^2 - 2\pi^2(L - d)]}{2\beta t_2[R(L - d) - (L - \beta)]} 
\]  
(17)

From which we get:

\[
 d = \frac{L(R + 1) - b}{R} \pm \sqrt{\frac{(L - b)^2}{R^2} + \frac{(L - b)^2 + \alpha^2}{R} + \beta^2}, 
\]  
(18)

for points of maxima or minima.
The positive value of the radical gives an imaginary value to $t_2$ since it throws $d$ outside of the limit assigned by (14). The negative value of the radical gives a minimum for positive values of $t_2$, and as before it is seen that there is but one such real point.

It is interesting to note that for these minimum points on the two curves the values of $t_1$ and $t_2$ are given by the expressions

$$t_1 = \pi \sqrt{\frac{2d}{g}} \quad \text{and} \quad t_2 = \pi \sqrt{\frac{2(L - d)}{g}},$$

thus showing that at these points the equivalent lengths are respectively equal to $2d$ and $2(L - d)$. If in addition to this we specify that $t_1 = t_2$ then it is seen that $L - d = d$, or $d = \frac{1}{2}L$. This involves the condition that the two curves shall be tangent to each other on the horizontal line

$$t = \pi \sqrt{\frac{L}{g}}.$$

An interesting case arises when $b = \frac{1}{2}L$. Equations (16) and (18) become

$$d' = -\frac{L}{2R} \pm \sqrt{\frac{L^2}{4R^2} + \frac{I^2 + 4a^2}{4R^2} + \beta^2},$$

$$d'' = L + \frac{L}{2R} \pm \sqrt{\frac{L^2}{4R^2} + \frac{I^2 + 4a^2}{4R^2} + \beta^2}.$$  

The negative value of the radical in the upper equation is precluded by equation (13), as is also the positive value in the lower equation by (14). Applying this fact, it will now be seen that $d' + d'' = L$, or in words: *when the centroid of the fixed part of the pendulum is placed midway between the knife-edges then the sum of the abscissas of the minimum points of the two curves equals the distance between the knife-edges.*

Of the nine possible intersections of two cubic curves, in the present case three are imaginary or at infinity, three belong to the condition that $t$ is negative, and three belong to positive values of $t$, and can hence be experimentally realized.

Subtracting equation (10) from (11) the locus of the line passing through the intersection points is obtained. This gives:
\[
\left( L - \frac{gR^2}{\pi^2} \right) \left[ L - 2b + R(L - 2d) \right] = 0.
\]

From this two straight lines are obtained. The first is the line

\[
t = \pi \sqrt{\frac{L}{g}}.
\]

and inspection shows that two of the intersections lie on this line which is parallel to the \(X\)-axis and at a distance \(\pi \sqrt{L/g}\) above it. It is one or other of these intersections (see Fig. 4) that is generally
sought for in experimental work and is the condition used by Kater, i.e., that the equivalent length of the pendulum shall equal $L$. Eliminating $t$ from equations (10) and (22) the values of the abscissas of the intersection points are had. This gives

$$d = \frac{1}{2}L \pm \sqrt{L^2 - 4\left[\frac{a^2}{R} + \beta^2 - \frac{b}{R}(L - b)\right]}.$$  \hspace{1cm} (23)

This expression shows that the intersection points lie symmetrically with respect to the point midway between $k_1$ and $k_x$. As seen above, the curves became tangent, i.e., the intersection points coincide at the point $(\frac{1}{2}L, \pi\sqrt{L/G})$. Under this condition the radical in (23) is equal to zero and

$$R(\beta^2 - \frac{1}{2}L^2) = b(L - b) - a^2.$$ \hspace{1cm} (24)

The second line obtained from (21) is:

$$d = \frac{L(R + 1) - 2b}{2R}.$$ \hspace{1cm} (25)

This line is parallel to the axis of $Y$ and passes through the third intersection of the two curves. This third intersection marks the condition when the centroid of the whole pendulum falls midway between $k_1$ and $k_x$. It may fall on either side of the first two intersections or between them. Again, it may coincide with either of them or all three may coincide. Substituting the value of $d$ in (25) into equation (10) the value of $t$ is found to be:

$$t = \left[\frac{\pi^2}{2g} \left(\frac{L - 2b + RL^2}{LR} + \frac{a^2 + R\beta^2}{R(R + 1)}\right)\right]^\frac{1}{2}.$$ \hspace{1cm} (26)

This is the common period from either knife-edge on the condition that $x_1 = x_x$. Whether this intersection fall below, upon, or above the line

$$t = \pi \sqrt{\frac{L}{g}}$$

depends upon whether

$$\frac{(L - 2b)^2}{R} + \frac{4(a^2 + R\beta^2)}{1 + R}$$

is less than, equal to, or greater than $L^2$. The coordinates of the three intersection points are as follows:
\[
\begin{align*}
\begin{cases}
  d_1 &= \frac{1}{2}L + \sqrt{RL^2 + 4(L - b)b - 4(a^2 + R\beta^2)} \\
  t_1 &= \pi \sqrt{\frac{L}{g}},
\end{cases} \\
\begin{cases}
  d_2 &= \frac{1}{2}L - \sqrt{RL^2 + 4(L - b)b - 4(a^2 + R\beta^2)} \\
  t_2 &= \pi \sqrt{\frac{L}{g}},
\end{cases} \\
\begin{cases}
  d_3 &= \frac{L(R + 1) - 2b}{2R} \\
  t_3 &= \left[\frac{\pi^2 (L - 2b + RL)^2 + 4R(\beta^2 + a^2 + R\beta^2)}{LR(R + 1)}\right]^{\frac{1}{3}}.
\end{cases}
\end{align*}
\] 

(27)

The conditions required for a coincidence of any two of the points are readily had by assuming the coordinates of 1 and 2, 1 and 3, and 2 and 3 successively equal. This gives three cases with a fourth where all three points coincide.

Case 1. \(d_1 = d_3 = \frac{1}{2}L\). — The term under the radical here equals zero. Setting this equal to zero and solving for \(b\) the expression is had

\[
  b = \frac{L}{2} \pm \sqrt{\frac{1}{4}L^3(R + 1) - a^2 - R\beta^2}.
\] 

(28)

These values of \(b\) are seen to be symmetrical with respect to the value \(d = \frac{1}{2}L\). If \(b = \frac{1}{2}L\) then the centroid of the whole pendulum is at this point (since \(d = \frac{1}{2}L\)) and the three intersections coincide. Keeping the first two intersections together, the third will move to the right or left according as the + or − sign holds good in equation (28).

Cases 2 and 3. — The condition for these cases may be expressed by the equation

\[
  L = 2b \pm 2R\sqrt{K}
\]

where

\[
  K = \frac{RL^2 + 4(L - b)b - 4(a^2 + R\beta^2)}{4R}.
\]

Solving for the value of \(b\) we find that
\[ b = \frac{1}{4}L \pm \sqrt{\frac{RL(R + 1) - 4R(a^2 + R\beta^2)}{R + 1}} \]  

(29)

The negative sign is to be used for case 2 and the positive sign for case 3. It is here seen that, beginning with the value \( b = \frac{1}{4}L \) (when all three intersections coincide) the tangency point recedes to the right (case 3) or left (case 2) as \( b \) increases or decreases. It will be noticed that a given change in the variable in equation (29) will produce an equal change in the value of \( b \) under both case 2 and case 3. In a sense, therefore, the points of tangency may be said to be symmetrically located with respect to the value \( b = \frac{1}{4}L \).

Case 4. — The condition reduces to the condition that \( d = \frac{1}{2}L \) and also \( b = \frac{1}{4}L \). This gives a value for \( R, R = (4a^2 - L^2)/(L^2 - 4\beta^2) \). Since \( R \) must be positive, this gives \( a \approx L/2 \approx \beta \). If \( R = 0, L^2 = 4a^2 \), a result given by equation (6) if \( L = l \). Finally it may be noted that if \( b = \frac{1}{4}L \) (but \( L - 2b = 2R\sqrt{K} \neq 0 \) then the third intersection falls on the line \( d = \frac{1}{2}L \) but below the line \( t = \pi\sqrt{L/g} \).

In the literature of the subject it is generally assumed that the centroid of the pendulum must not be allowed to fall midway between \( k_1 \) and \( k_2 \). From the preceding it is seen that this is permissible and that the value of \( t \) is then given by equation (26); and, further, that this expression may reduce to the usual one given in equation (22). A glance at equation (26) is sufficient, however, to convince one that this equation cannot be used in practice; also, when the three points of intersection coincide and equation (26) reduces to (22) the curves become osculating curves and the intersection point is not well defined, hence the condition \( \varepsilon_1 = \varepsilon_2 \) is in practice on all accounts to be avoided.

**Division II. \( \beta \) and \( \beta \) Taken as Variables.**

In order that a change in \( \beta \) shall not produce a change in \( d \) also, the pendulum must be modified as shown in Fig. 5, which represents a pendulum with two knife-edges \( k_1 \) and \( k_2 \) and with two movable masses \( m_1, m_2 \) which can be so moved along the bar of the pendulum that their centroid \( \varepsilon_3 \) shall remain fixed. The distance of these masses from \( \varepsilon_3 \) may be called \( g \) and it is seen that the moment of inertia of the movable mass becomes \( 2m\beta_0^2 + 2mg^2 \) in
which \( m\tilde{p}_0^2 \) is the moment of inertia of \( m_0 \), \( m_0 \), each about its own centroid. Equation (10) may now be written

\[
gt^2/\pi^2(b + Rd) = Rq^2 + R\tilde{p}_0^2 + Rd^2 + b^2 + a^2. \tag{30}
\]

\( t \) and \( q \) are the variables in this equation and it is therefore of the form

\[
Aq^2 = Bq^2 + C \tag{31}
\]

where

\[
A = q/\pi^2(b + Rd), \quad B = R, \quad \text{and} \quad C = R(\tilde{p}_0^2 + d^2) + b^2 + a^2.
\]

The constants \( A, B, \) and \( C \) are necessarily positive.

Equation (31) is the equation of a central hyperbola whose axis is the axis of \( Y \). Since negative values of \( t \) and \( q \) are not permissible, that part of the curve falling in the first quadrant is alone to be considered. From equation (31) it appears that \( t \) is a minimum for \( q \) equals zero, varies in the same sense as \( C \) and in the opposite sense as \( A \). The equation of the asymptotes is,

\[
t = \pm \frac{B}{A} q = \pm \frac{R\pi^2 q}{g(\tilde{p} + Rd)}, \tag{32}
\]

which shows that the slope of the asymptotes varies inversely as \( \left(\frac{b}{R + d}\right) \).

If the pendulum be swung from the second knife-edge equation (27) becomes:

\[
g\tilde{p}^2_0 \left[ L - b + R(L - d) \right] = Rq^2 + R\tilde{p}_0^2 + R(L - d)^2 + (L - b)^2 + a^2 \tag{33}
\]

which is of the same form as (28) but with changed values for the constants.

If in equations (27) and (30) \( y \) be substituted for \( t^2 \) and \( x \) for \( q^2 \) as the variables, the equations take the form,

\[
Ay = Bx + C \quad \text{and} \quad A'y = B'x + C' \tag{34}
\]

in which \( A, B, \) and \( C \) and \( A', B', \) and \( C' \) have the same values as before. Equations (34) are the equations of two straight lines.
having different slopes. The intersections of all the pairs of lines represented by (34) lie on the line,

\[ y = \frac{C - C'}{A - A'} = \frac{\pi^2 L}{g}. \]  

(35)

The condition for a realizable intersection is that the coordinates \((x_i, y_i)\) of the point of intersection both be positive. \(y_i\) necessarily is positive, from (35); and substituting this value of \(y_i\) in the equations of (34) shows that \(x_i\) is positive, provided

\[ L > (I_e + I_{sm} + M\theta^2 + 2md^2)/(M\theta + 2md). \]

This condition may be expressed thus: \(x\) is positive when the distance between the knife-edges is greater than the ratio between the moment of inertia of the whole pendulum about the supporting axis, and its moment about the same axis.

If the slope of the lines of equation (34) be \(\tan \varphi_1\) and \(\tan \varphi_2\), then

\[ \tan \varphi_1 = \frac{\pi^2 R}{g(b + Rd)} = \frac{2\pi^2 m}{g(M + 2m)}, \]

and

\[ \tan \varphi_2 = \frac{\pi^2}{g} \frac{R}{L(R + 1) - Rd - b} = \frac{\pi^2}{g} \frac{2m}{(L - z)(M + 2m)}. \]

In order that the intersection of the lines be as well defined as possible, the angle \((\varphi_1 - \varphi_2)\) must be as large as possible. So long as at least one of the angles is small this condition may be stated as \((\tan \varphi_1 - \tan \varphi_2)\) as great as possible. This leads to the expression:

\[ \tan \varphi_1 - \tan \varphi_2 = \frac{2\pi^2 m}{g(M + 2m)} \left( \frac{1}{z} - \frac{1}{L - z} \right). \]  

(36)

This means \(m\) large, \(M\) small, \(i.e., R\) large, \(z\) small and \(L\) large. Under these conditions (36) practically reduces to

\[ \tan \varphi_1 - \tan \varphi_2 = \frac{\pi^2}{g} \times \frac{1}{z}, \]

and is therefore inversely proportional to \(z\). Now approximately \(z = b/R + d\), and the above condition is seen to be that \(R\) shall be
large, \( b \) and \( d \) small and one of them negative: it being remembered that \( z \) must remain finite and positive.

Such a pendulum would consist of a bar as light as rigidity would permit, carrying two knife-edges and having two equal movable masses. Such a pendulum is shown in Fig. 5 and data derived from it illustrated in Fig. 6. In the lower diagram of this figure the intersection of the two lines is obtained by extrapolation and by using a pendulum so proportioned that it is impossible to bring it into final adjustment. It is a curious fact that if the graphs of the hyperbolæ themselves had been plotted (as in the first division of the paper) no intersection would have been secured. This is evident for the required value of \( q^2 \) is negative and \( q \) therefore imaginary. The curves would simply approach and recede as do curves
II. in Fig. 4. In spite of this fact the true value of $g$ is gotten from the data, and the point of intersection in Fig. 6 is the means for getting this value.

**Division III. $t$ and $R$ taken as variables.**

If in equations (10) and (11) $t$ and $R$ be taken as the variables and the origin of (10) be moved to $(-b/d, 0)$ and of (11) to $-(L - b)(L - d)$, $0$ these equations reduce to:

$$
\begin{align*}
  t_1^2 R' &= AR' + B \\
  t_2^2 R' &= A'R' + B'
\end{align*}
$$

where

$$
A = \frac{\pi^2}{g} \frac{d^2 + \beta^2}{d}, \quad B = \frac{\pi^2}{g} \frac{b}{d} \left( \frac{b^2 + a^2}{b} - \frac{d^2 + \beta^2}{d} \right),
$$

$$
A' = \frac{\pi^2}{g} \frac{(L - d)^2 + \beta^2}{L - d}, \quad B' = \frac{\pi^2}{g} \frac{L - b}{L - d} \left( \frac{(L - b)^2 + a^2}{L - b} - \frac{(L - d)^2 + \beta^2}{L - d} \right).
$$

If the constants be positive these equations belong to Newton's sixtieth species of cubics. If $B$ or $B'$ be negative while the coeffi-

![Fig. 7.](image)

cient of $R'$ is positive, changing $R'$ to $-R'$ will have the effect of changing the sign of the absolute term, thus reducing the equa-
tion to the same species as before. If \( A \) or \( A' \) be negative, the corresponding equation will belong to Newton's sixty-third species, after changing \( R' \) to \(- R'\) if necessary, to make the absolute term positive.

In equation (10) four forms arise (Fig. 7). (A) \( b \) and \( d \) positive and \((\beta^2 + \alpha^2)/b > (d^3 + \beta^2)/d\). This makes \( A \) and \( B \) positive. (B) \( b \) and \( d \) positive, and \((\beta^2 + \alpha^2)/b < (d^3 + \beta^2)/d\). This makes \( A \) positive and \( B \) negative. (C) \( b \) negative and \( d \) positive. This makes both \( A \) and \( B \) positive. (D) \( b \) positive and \( d \) negative. This makes \( A \) and \( B \) both negative. Case (A) is distinguished from (C) in that the asymptote \( R = -b/d \), to equation (10) is on the negative side of the \( Y \)-axis in (A) and on the positive side in (C). The values \( b \) and \( d \) negative, gives two more cases, which, however, do not concern us as no part of the resulting curves lies in the first quadrant. The four cases are illustrated in Fig. 7, where the general shape of the resulting curves is shown. Equation (11) yields a similar number of cases by substituting \( L - b \) for \( b \) and \( L - d \) for \( d \).

Differentiating (10) and (11) with respect to \( R \) we obtain,

\[
\begin{align*}
\frac{dt}{dR} &= \pi \left( \frac{(d^3 + \beta^2)b - (\beta^2 + \alpha^2)d}{2\sqrt{g} (Rd + b)^4 \left[ (d^2 + \beta^2)R + \beta^2 + \alpha^2 \right]^4} \right) \\
\frac{dt'}{dR} &= \pi \left[ (L-d)^2 + \beta^2 \right] (L-b) - \left[ (L-b)^2 + \alpha^2 \right] (L-d) \\
\frac{dR}{dR} &= \frac{(Rd + b)^4 \left[ (d^2 + \beta^2)R + \beta^2 + \alpha^2 \right]^4}{2\sqrt{g} [R(L-d) + L-b]^4 \left[ (L-d)^2 + \beta^2 \right] R + (L-b)^2 + \alpha^2} \\
\end{align*}
\]

The form of these derivatives shows that the curves have on maxima or minima. If \( b \) and \( d \) are both positive, \( dt/dR \) is always positive or always negative according as \((d^3 + \beta^2)/d > (\beta^2 + \alpha^2)/\beta\). When

\[ R = 0, \quad t = \pi \sqrt{\frac{\beta^2 + \alpha^2}{gb}}. \]

When

\[ R = \infty, \quad t = \pi \sqrt{\frac{d^2 + \beta^2}{gd}}. \]

The corresponding conclusions are reached for \( dt'/dR \) by putting \( L - b \) for \( b \) and \( L - d \) for \( d \).

Equation (21) gives the lines passing through the intersection points, as in Division II., and the following are the coordinates of the intersections. From the first factor:
\[ t_1 = \pm \pi \sqrt{\frac{L}{g}}. \]
\[ R_1 = -\frac{Lb - (\beta^2 + a^2)}{Ld - (d^2 + \beta^2)} \]

From the second factor:
\[ t_2 = \pm \frac{\pi}{\sqrt{g}} \left[ \frac{(L - 2d)(\beta^2 + a^2) - (L - 2b)(d^2 + \beta^2)}{L(b - d)} \right]^{\frac{1}{4}} \]
\[ R_2 = -\frac{L - 2b}{L - 2d}. \]

It is here seen that while \( t \) is double valued (the + value alone being available) \( R \) is single valued in both cases. There are thus but two intersection points that can fall within the first quadrant.

Of course, the condition realized by the first intersection is the one desired, since here equation (22) holds good. To make it possible to obtain this intersection \( R_1 \) must be positive, and the following table gives all combinations of the signs of the four quantities \( b, L - b, d, L - d \), with which it is possible to effect this. To the right of each combination are the necessary accompanying conditions on the constants. It will be found that all other combinations are either mere duplicates formed by interchanging the notation with respect to the knife-edges, or else are impossible because \( L \) and \( R \) are necessarily positive.

<table>
<thead>
<tr>
<th>( b )</th>
<th>( L - b )</th>
<th>( d )</th>
<th>( L - d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I.</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>II.</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>III.</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>IV.</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

Combination I. produces curve \( A \) (Fig. 7) from both knife-edges; II. gives curve \( B \) from both knife-edges; III. gives curves \( B \) and \( D \) from \( k_1 \) and \( k_2 \) respectively; IV. gives curve \( A \) from \( k_1 \) and curve \( B \) from \( k_x \).
Conditions of the second intersection may be realized by any combination provided it is possible by a change in $R$ to throw the centroid of the whole pendulum midway between $k_1$ and $k_x$. Since a tangency point, or conditions approaching it are preferably avoided, it would seem likely that a better defined first intersection would be obtained by rendering the second intersection impossible. This may be secured by adjusting the constants so that $R_2$ in (40) will be negative. The value of $R_2$ may be written

$$R_2 = -\frac{(L - b) - b}{(L - a) - d},$$

(41)

a form that shows the symmetrical occurrence of the four determining quantities. In cases I and II, where these quantities are all positive, in order to have $R_2$ negative $\frac{1}{2}L$ must be either greater or less than both $b$ and $d$. If $\frac{1}{2}L$ be less than both, the conditions under case I, give $d > \beta$ and $L - b < a$; if $\frac{1}{2}L$ be greater than both, $b < a$ and $L - d > \beta$, for a positive first intersection. In case II, if $\frac{1}{2}L$ be less than $b$ and $d$, we have $d > a$ and $L - d < \beta$; if $\frac{1}{2}L$ be greater, then $d < \beta$ and $L - b > a$. In case III, the necessary and sufficient condition for $R_1$ positive and $R_2$ negative is

$$2b > L > \frac{b^2 + a^2}{\beta}.$$

This shows that we must have $b > a$ and consequently $b > \frac{1}{2}L > a$, a simpler condition but not so precise as the preceding. The condition in case IV, is

$$2d > L > \frac{d^2 + \beta^2}{d},$$

from which may be deduced the simpler condition $d > \frac{1}{2}L > \beta$.

If we place the values of $R_1$ and $t_1$ from (39) in the expressions for the slopes of the two curves, given in (38) we get

$$\frac{dt_1}{dR_1} = \frac{\pi}{2\sqrt{Lg}} \frac{(a^2 + \beta^2 - Ld)^2}{b(d^2 + \beta^2) - d(b^2 + a^2)},$$

(42)

$$\frac{dt_1'}{dR_1} = \frac{\pi}{2\sqrt{Lg}} \frac{(d^2 + \beta^2 - Ld)^2}{(L - b)[(L - d)^2 + \beta^2] - (L - d)[(L - b)^2 + a^2]},$$

(43)
These expressions will be either both positive or both negative under the conditions we are discussing, hence, in general, to get a good angle of intersection it is well to make one as large, and the other as small, as practicable. To make an approximation we may make the denominator of (42) equal to zero, though in practice it should not exactly equal zero as then the curves break down into straight lines and the point of intersection \( R \) becomes indeterminate. Doing this, the denominator of (43) reduces to

\[
L[(b - d)(L - b - d) - a^2 + \beta^2].
\]

This should be large in order to make (43) small. To attain this we must evidently have the difference between \( b \) and \( d \), and also the difference between \( a \) and \( \beta \), large.

The above analysis, while reducing the number of permissible cases to four, does not furnish a sufficient criterion between them.

Fig. 8.

It is seen that any of the four may be realized experimentally and it would seem best to leave the question as to which (if any) of the four is to be preferred to be decided by experiment. A design for a pendulum meeting the requirements is given in Fig. 8. \( R \) is to be varied by adding thin, circular plates of metal to the cross-piece. It will be noticed that by using this form \( \beta \) is kept constant, as lengthening the cylinder does not alter its radius of gyration about its own axis.

We wish to acknowledge the assistance of Mr. Wm. N. Birchby, particularly in the preparation of the third division. Our thanks are also due to Mr. M. D. Hersey for the preparation of the diagrams.

Physical Laboratory,
Colorado College,
December 1, 1906