Translation of Einstein’s Attempt of a Unified Field Theory with Teleparallelism

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Abstract

We present the first English translation of Einstein’s original papers related to the teleparallel\(^1\) attempt of an unified field theory of gravitation and electromagnetism. Our collection contains the summarizing paper in Math. Annal. 102 (1930) pp. 685-697 and 2 reports published in ‘Sitzungsberichte der Preussischen Akademie der Wissenschaften’ on June 7th, 1928 (pp. 217-221), June 14th, 1928 (pp. 224-227) and a precursor report (July 9th, 1925 pp. 414-419). To ease understanding, literature on tensor analysis is quoted in the footnotes.

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\(^1\)‘absolute parallelism’, ‘distant parallelism’ and the German ‘Fernparallelismus’ are synonyms.
Among the theoretical physicists working in the field of the general theory of relativity there should be a consensus about the consubstantiality of the gravitational and electromagnetic field. However, I was not able to succeed in finding a convincing formulation of this connection so far. Even in my article published in these session reports (XVII, p. 137, 1923) which is entirely based on the foundations of EDDINGTON, I was of the opinion that it does not reflect the true solution of this problem. After searching ceaselessly in the past two years I think I have now found the true solution. I am going to communicate it in the following.

The applied method can be characterized as follows. First, I looked for the formally most simple expression for the law of gravitation in the absence of an electromagnetic field, and then the most natural generalization of this law. This theory appeared to contain MAXWELL’s theory in first approximation.

In the following I shall outline the scheme of the general theory (§ 1) and then show in which sense this contains the law of the pure gravitational field (§ 2) and MAXWELL’s theory (§ 3).

§ 1. The general theory

We consider a 4-dimensional continuum with an affine connection, i.e. a $\Gamma^\mu_{\alpha\beta}$-field which defines infinitesimal vector shifts according to the relation

$$dA^\mu = -\Gamma^\mu_{\alpha\beta} A^\alpha dx^\beta. \quad (1)$$

We do not assume symmetry of the $\Gamma^\mu_{\alpha\beta}$ with respect to the indices $\alpha$ and $\beta$. From these quantities $\Gamma$ we can derive the Riemannian tensors

$$R^\alpha_{\mu.\nu\beta} = -\frac{\partial \Gamma^\alpha_{\mu\nu}}{\partial x^\beta} + \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\mu\beta} + \frac{\partial \Gamma^\alpha_{\mu\beta}}{\partial x^\nu} + \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\mu\beta},$$

and

$$R_{\mu\nu} = R^\alpha_{\mu.\nu\alpha} = -\frac{\partial \Gamma^\mu_{\alpha\nu}}{\partial x^\alpha} + \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\alpha\nu} + \frac{\partial \Gamma^\mu_{\alpha\sigma}}{\partial x^\nu} + \Gamma^\mu_{\alpha\sigma} \Gamma^\sigma_{\alpha\beta} \quad (2)$$

in a well-known manner. Independently from this affine connection we introduce a contravariant tensor density $g^{\mu\nu}$, whose symmetry properties we leave undetermined as well. From both quantities we obtain the scalar density

$$\mathcal{S} = g^{\mu\nu} R_{\mu\nu} \quad (3)$$

and postulate that all the variations of the integral

$$\mathcal{J} = \int \mathcal{S} dx^1 dx^2 dx^3 dx^4$$
with respect to the $g_{\mu\nu}$ and $\Gamma_{\mu\nu}^\alpha$ as independent (i.e. not to be varied at the boundaries) variables vanish.

The variation with respect to the $g_{\mu\nu}$ yields the 16 equations

$$R_{\mu\nu} = 0,$$

the variation with respect to the $\Gamma_{\mu\nu}^\alpha$ at first the 64 equations

$$\frac{\partial g_{\mu\nu}}{\partial x_\alpha} + g_{\beta\nu} \Gamma_{\mu\beta}^{\alpha} + g_{\mu\beta} \Gamma_{\alpha\beta}^{\nu} - \delta_\alpha^{\nu} \left( \frac{\partial g_{\mu\beta}}{\partial x_\beta} + g_{\sigma\beta} \Gamma_{\sigma\beta}^{\mu} \right) - g_{\mu\nu} \Gamma_{\alpha\beta} = 0.$$  

(5)

We are going to begin with some considerations that allow us to replace the eqns. (5) by simpler ones.

If we contract the l.h.s. of (5) by $\nu$ and $\alpha$ or $\mu$ and $\alpha$, we obtain the equations

$$3 \left( \frac{\partial g_{\mu\alpha}}{\partial x_\alpha} + g_{\alpha\beta} \Gamma_{\mu\beta}^{\alpha} \right) + g_{\mu\alpha} \left( \Gamma_{\beta\alpha}^{\beta} - \Gamma_{\beta\alpha}^{\beta} \right) = 0.$$  

(6)

$$\frac{\partial g_{\mu\alpha}}{\partial x_\alpha} - \frac{\partial g_{\alpha\nu}}{\partial x_\alpha} = 0.$$  

(7)

If we further introduce the quantities $g_{\mu\nu}$ which are the normalized subdeterminants of the $g_{\mu\nu}$ and thus fulfill the equations

$$g_{\mu\alpha} g_{\nu\alpha} = g_{\alpha\mu} g_{\alpha\nu} = \delta_\nu^{\alpha},$$

and if we now multiply (5) by $g_{\mu\nu}$, after pulling up one index the result may be written as follows:

$$2 g_{\mu\alpha} \left( \frac{\partial \log g}{\partial x_\alpha} + \Gamma_{\alpha\beta}^{\beta} \right) + g_{\alpha\beta} (\Gamma_{\beta\alpha}^{\beta} - \Gamma_{\beta\alpha}^{\beta}) + \delta_\mu^{\nu} \left( \frac{\partial g_{\beta\alpha}}{\partial x_\alpha} + g_{\sigma\alpha} \Gamma_{\sigma\alpha}^{\beta} \right) = 0.$$  

(8)

while $\gamma$ denotes the determinant of $g_{\mu\nu}$. The equations (6) and (8) we write in the form

$$f^\mu = \frac{1}{3} g_{\mu\alpha} \left( \Gamma_{\alpha\beta}^{\beta} - \Gamma_{\beta\alpha}^{\beta} \right) = - \left( \frac{\partial g_{\mu\alpha}}{\partial x_\alpha} + g_{\alpha\beta} \Gamma_{\mu\beta}^{\alpha} \right) = - g_{\mu\alpha} \left( \frac{\partial \log g}{\partial x_\alpha} + \Gamma_{\alpha\beta}^{\beta} \right),$$

(9)

whereby $f^\mu$ stands for a certain tensor density. It is easy to prove that the system (5) is equivalent to the system

$$\frac{\partial g_{\mu\nu}}{\partial x_\alpha} + g_{\beta\nu} \Gamma_{\mu\beta}^{\alpha} + g_{\mu\beta} \Gamma_{\alpha\beta}^{\nu} - g_{\mu\nu} \Gamma_{\alpha\beta}^{\beta} + \delta_\nu^{\mu} f^\alpha = 0$$

(10)

in conjunction with (7). By pulling down the upper indices we obtain the relations

$$g_{\mu\nu} = g_{\mu\nu} = g_{\mu\nu} \sqrt{-g},$$

whereby $g_{\mu\nu}$ is a covariant tensor

$$- \frac{\partial g_{\mu\nu}}{\partial x_\alpha} + g_{\sigma\nu} \Gamma_{\mu\alpha}^{\sigma} + g_{\mu\sigma} \Gamma_{\alpha\nu}^{\sigma} + g_{\mu\nu} \phi_\alpha + g_{\mu\sigma} \phi_\nu = 0,$$

(10\ a)

whereby $\phi_\tau$ is a covariant vector. This system, together with the two systems given above,

$$\frac{\partial g_{\mu\nu}}{\partial x_\alpha} - \frac{\partial g_{\alpha\nu}}{\partial x_\alpha} = 0$$

(7)

and

$$0 = R_{\mu\nu} = - \frac{\partial \Gamma_{\mu\alpha}^{\alpha}}{\partial x_\alpha} + \Gamma_{\mu\beta}^{\alpha} \Gamma_{\alpha\nu}^{\beta} + \frac{\partial \Gamma_{\alpha\nu}^{\alpha}}{\partial x_\nu} - \Gamma_{\mu\nu}^{\alpha} \Gamma_{\alpha\beta}^{\beta},$$

(4)
are the result of the variational principle in the most simple form. Looking at this result, it is remarkable that the vector \( \phi_\tau \) occurs besides the tensor \( g_{\mu\nu} \) and the quantities \( \Gamma^\alpha_{\mu\nu} \). To obtain consistency with the known laws of gravitation and electricity, we have to interpret the symmetric part of \( g_{\mu\nu} \) as metric tensor and the skew-symmetric part as electromagnetic field, and we have to assume the vanishing of \( \phi_\tau \), which will be done in the following. For a later analysis (e.g. the problem of the electron), we will have to keep in mind that the Hamiltonian principle does not indicate a vanishing \( \phi_\tau \). Setting \( \phi_\tau \) to zero leads to an overdetermination of the field, since we have 16 + 64 + 4 algebraically independent differential equations for 16 + 64 variables.

§ 2. The pure gravitational field as special case

Let the \( g_{\mu\nu} \) be symmetric. The equations (7) are fulfilled identically. By changing \( \mu \) to \( \nu \) in (10a) and subtraction we obtain in easily understandable notation

\[
\Gamma_{\nu,\mu\alpha} + \Gamma_{\mu,\alpha\nu} - \Gamma_{\mu,\nu\alpha} - \Gamma_{\nu,\alpha\mu} = 0. \tag{11}
\]

If \( \Delta \) is called the skew-symmetric part of \( \Gamma \) with respect to the last two indices, (11) takes the form

\[
\Delta_{\nu,\mu\alpha} + \Delta_{\mu,\alpha\nu} = 0
\]
or

\[
\Delta_{\nu,\mu\alpha} = \Delta_{\mu,\nu\alpha}. \tag{11a}
\]

This symmetry property of the first two indices contradicts the antisymmetry of the last ones, as we learn from the series of equations

\[
\Delta_{\mu,\nu\alpha} = -\Delta_{\mu,\alpha\nu} = -\Delta_{\alpha,\mu\nu} = \Delta_{\alpha,\nu\mu} = \Delta_{\nu,\alpha\mu} = -\Delta_{\nu,\mu\alpha}.
\]

This, in conjunction with (11a), compels the vanishing of all \( \Delta \). Therefore, the \( \Gamma \) are symmetric in the last two indices as in RIEMANNian geometry. The equations (10a) can be resolved in a well-known manner, and one obtains

\[
\frac{1}{2} g_{\alpha\beta} \left( \frac{\partial g_{\mu\beta}}{\partial x_\nu} + \frac{\partial g_{\nu\beta}}{\partial x_\mu} - \frac{\partial g_{\mu\nu}}{\partial x_\beta} \right). \tag{12}
\]

Equation (12), together with (4) is the well-known law of gravitation. Had we presumed the symmetry of the \( g_{\mu\nu} \) at the beginning, we would have arrived at (12) and (4) directly. This seems to be the most simple and coherent derivation of the gravitational equations for the vacuum to me. Therefore it should be seen as a natural attempt to encompass the law of electromagnetism by generalizing these considerations rightly. Had we not assumed the vanishing of the \( \phi_\tau \), we would have been unable to derive the known law of the gravitational field in the above manner by assuming the symmetry of the \( g_{\mu\nu} \). Had we assumed the symmetry of both the \( g_{\mu\nu} \) and the \( \Gamma^\alpha_{\mu\nu} \) instead, the vanishing of \( \phi_\alpha \) would have been a consequence of (9) or (10a) and (7); we would have obtained the law of the pure gravitational field as well.

§ 3. Relations to Maxwell’s theory

If there is an electromagnetic field, that means the \( g^\mu\nu \) or the \( g_{\mu\nu} \) do contain a skew-symmetric part, we cannot solve the eqns. (10a) any more with respect to the \( \Gamma^\alpha_{\mu\nu} \), which significantly complicates the clearness of the whole system. We succeed in resolving the problem however, if we restrict ourselves to the first approximation. We shall do this and once again postulate the vanishing of \( \phi_\tau \). Thus we start with the ansatz

\[
g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu} + \phi_{\mu\nu}, \tag{13}
\]
whereby the $\gamma_{\mu\nu}$ should be symmetric, and the $\phi_{\mu\nu}$ skew-symmetric, both should be infinitely small in first order. We neglect quantities of second and higher orders. Then the $\Gamma_{\alpha}^{\mu\nu}$ are infinitely small in first order as well.

Under these circumstances the system (10a) takes the more simple form

$$+ \frac{\partial g^{\mu\nu}}{\partial x_\alpha} + \Gamma_\alpha^\nu + \Gamma_\alpha^\mu = 0. \quad (10b)$$

After applying two cyclic permutations of the indices $\mu$, $\nu$, and $\alpha$ two further equations appear. Then, out of the three equations we may calculate the $\Gamma_{\alpha}^{\mu\nu}$ in a similar manner as in the symmetric case. One obtains

$$-\Gamma_{\alpha}^{\mu\nu} = \frac{1}{2} \left( \frac{\partial g_{\alpha\nu}}{\partial x_\mu} + \frac{\partial g_{\mu\alpha}}{\partial x_\nu} - \frac{\partial g_{\nu\mu}}{\partial x_\alpha} \right). \quad (14)$$

Eqn. (4) is reduced to the first and third term. If we put the expression $\Gamma_{\alpha}^{\mu\nu}$ from (14) therein, one obtains

$$-\frac{\partial^2 g_{\alpha\mu}}{\partial x_\alpha^2} + \frac{\partial^2 g_{\alpha\nu}}{\partial x_\nu^2} + \frac{\partial^2 g_{\alpha\nu}}{\partial x_\mu^2} \frac{\partial g_{\nu\mu}}{\partial x_\alpha} - \frac{\partial^2 g_{\alpha\alpha}}{\partial x_\mu \partial x_\nu} = 0. \quad (15)$$

Before further consideration of (15), we develop the series from equation (7). Firstly, out of (13) follows that the approximation we are interested in yields

$$g^{\mu\nu} = -\delta_{\mu\nu} - \gamma_{\mu\nu} + \phi_{\mu\nu}, \quad (16)$$

Regarding this, (7) transforms to

$$\frac{\partial \phi_{\mu\nu}}{\partial x_\nu} = 0. \quad (17)$$

Now we put the expressions given by (13) into (15) and obtain with respect to (17)

$$-\frac{\partial^2 \gamma_{\mu\nu}}{\partial x_\alpha^2} + \frac{\partial^2 \gamma_{\mu\alpha}}{\partial x_\nu \partial x_\alpha} + \frac{\partial^2 \gamma_{\nu\alpha}}{\partial x_\mu \partial x_\alpha} - \frac{\partial^2 \gamma_{\alpha\alpha}}{\partial x_\mu \partial x_\nu} = 0 \quad (18)$$

$$\frac{\partial^2 \phi_{\mu\nu}}{\partial x_\alpha^2} = 0. \quad (19)$$

The expressions (18), which may be simplified as usual by proper choice of coordinates, are the same as in the absence of an electromagnetic field. In the same manner, the equations (17) and (19) for the electromagnetic field do not contain the quantities $\gamma_{\mu\nu}$ which refer to the gravitational field. Thus both fields are - in accordance with experience - independent in first approximation.

The equations (17), (19) are nearly equivalent to MAXWELL’s equations of empty space. (17) is one MAXWELLian system. The expressions

$$\frac{\partial \phi_{\mu\nu}}{\partial x_\alpha} + \frac{\partial \phi_{\nu\alpha}}{\partial x_\mu} + \frac{\partial \phi_{\alpha\mu}}{\partial x_\nu},$$

which\footnote{This appears to be a misprint. The first term should be squared.} according to MAXWELL should vanish, do not vanish necessarily due to (17) and (19), but their divergences of the form

$$\frac{\partial}{\partial x_\alpha} \left( \frac{\partial \phi_{\mu\nu}}{\partial x_\alpha} + \frac{\partial \phi_{\nu\alpha}}{\partial x_\mu} + \frac{\partial \phi_{\alpha\mu}}{\partial x_\nu} \right)$$

however do. Thus (17) and (19) are substantially identical to MAXWELL’s equations of empty space. Concerning the attribution of $\phi_{\mu\nu}$ to the electric and magnetic vectors (a and b) I would like to make a comment that claims validity independently from the theory presented here. According to classical mechanics that uses central forces to every sequence of motion $V$ there is an inverse motion $\bar{V}$, that
passes the same configurations by taking an inverse succession. This inverse motion $\bar{V}$ is formally obtained from $V$ by substituting

$$
x' = x \\
y' = y \\
z' = z \\
t' = -t
$$

in the latter one.

We observe a similar behavior, according to the general theory of relativity, in the case of a pure gravitational field. To achieve the solution $\bar{V}$ out of $V$, one has to substitute $t' = -t$ into all field functions and to change the sign of the field components $g_{14}, g_{24}, g_{34}$ and the energy components $T_{14}, T_{24}, T_{34}$. This is basically the same procedure as applying the above transformation to the primary motion $V$. The change of signs in $g_{14}, g_{24}, g_{34}$ and in $T_{14}, T_{24}, T_{34}$ is an intrinsic consequence of the transformation law for tensors.

This generation of the inverse motion by transformation of the time coordinate ($t' = -t$) should be regarded as a general law that claims validity for electromagnetic processes as well. There, an inversion of the process changes the sign of the magnetic components, but not those of the electric ones. Therefore one should have to assign the components $\phi_{23}, \phi_{31}, \phi_{12}$ to the electric field and $\phi_{14}, \phi_{24}, \phi_{34}$ to the magnetic field. We have to give up the inverse assignment which was in use as yet. It was preferred so far, since it seems more comfortable to express the density of a current by a vector rather than by a skew-symmetric tensor of third rank.

Thus in the theory outlined here, (7) respectively (17) is the expression for the law of magnetoelectric induction. In accordance, at the r.h.s. of the equation there is no term that could be interpreted as density of the electric current.

The next issue is, if the theory developed here renders the existence of singularity-free, centrally symmetric electric masses comprehensible. I started to tackle this problem together with Mr. J. GROMMER, who was at my disposal ceaselessly for all calculations while analyzing the general theory of relativity in the last years. At this point I would like to express my best thanks to him and to the ‘International educational board’ which has rendered possible the continuing collaboration with Mr. GROMMER.
Riemannian Geometry with Maintaining the Notion of Distant Parallelism

Albert Einstein

translation by A. Unzicker and T. Case

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Riemannian Geometry has led to a physical description of the gravitational field in the theory of general relativity, but it did not provide concepts that can be attributed to the electromagnetic field. Therefore, theoreticians aim to find natural generalizations or extensions of riemannian geometry that are richer in concepts, hoping to arrive at a logical construction that unifies all physical field concepts under one single leading point. Such endeavors brought me to a theory which should be communicated even without attempting any physical interpretation, because it can claim a certain interest just because of the naturality of the concepts introduced therein.

Riemannian geometry is characterized by an Euclidean metric in an infinitesimal neighborhood of any point \( P \). Furthermore, the absolute values of the line elements which belong to the neighborhood of two points \( P \) and \( Q \) of finite distance can be compared. However, the notion of parallelism of such line elements is missing; a concept of direction does not exist for the finite case. The theory outlined in the following is characterized by introducing - beyond the Riemannian metric- the concept of ‘direction’, ‘equality of directions’ or ‘parallelism’ for finite distances. Therefore, new invariants and tensors will arise besides those known in Riemannian geometry.

\[ \frac{1}{\text{n-bein field and metric}} \]

Given an arbitrary point \( P \) of the \( n \)-dimensional continuum, let’s imagine an orthogonal \( n \)-bein of \( n \) unit vectors that represents a local coordinate system. \( A_a \) are the components of a line element or another vector with respect to this local system (\( n \)-bein). Besides that, we introduce a Gaussian coordinate system of the \( x^\nu \) for describing a finite domain. Let \( A^\nu \) be the components of a vector (A) with respect to the latter, and \( h^\nu_a \) the \( \nu \)-components of the unit vectors forming the \( n \)-bein. Then, we have\(^3\)

\[ A^\nu = h^\nu_a A_a \ldots \]

(1)

One obtains the inversion of (1) by calling \( h^\nu_a \) the normalized subdeterminants of the \( h^\nu_a \),

\[ A_a = h_{\mu a} A^\mu \ldots \]  

(1a)

Since the infinitesimal sets are Euclidean,

\[ A^2 = \sum A_a^2 = h_{\mu a} h_{\nu a} A^\mu A^\nu \ldots \]

holds for the modulus \( A \) of the vector (A).

\(^3\)We assign Greek letters to the coordinate indices and Latin ones to the bein indices.
Therefore, the components of the metric tensor appear in the form

\[ g_{\mu\nu} = h_{\mu a} h_{\nu a}, \ldots \]  

whereby the sum has to be taken over \( a \). For a fixed \( a \), the \( h_{\mu a} \) are the components of a contravariant vector. Furthermore, the following relations hold:

\[ h_{\mu a} h_{\nu a} = \delta^\nu_\mu, \ldots \]  

(4)

\[ h_{\mu a} h_{\mu b} = \delta_{ab}, \ldots \]  

(5)

with \( \delta = 1 \) if the indices are equal, and \( \delta = 0 \), if not. The correctness of (4) and (5) follows from the above definition of the \( h_{\mu a} \) as the normalized subdeterminants of the \( h^a_\mu \). The vector property of \( h_{\mu a} \) follows conveniently from the fact that the l.h.s. and therefore, the r.h.s. of (1a) as well, are invariant for any coordinate transformation and for any choice of the vector \( (A) \). The \( n \)-bein field is determined by \( n^2 \) functions \( h^a_\mu \), whereas the Riemannian metric is determined just by \( \frac{n(n+1)}{2} \) quantities. According to (3), the metric is determined by the \( n \)-bein field but not vice versa.

§ 2. Teleparallelism and rotation invariance

By postulating the existence of the \( n \)-bein field (in every point) one expresses implicitly the existence of a Riemannian metric and distant parallelism. \( (A) \) and \( (B) \) being two vectors in the points \( P \) and \( Q \) which have the same local coordinates with respect to their \( n \)-beins (that means \( A_a = B_a \)), then have to be regarded as equal (because of (2)) and as 'parallel'. If we take the metric and the teleparallelism as the essential, i.e. the objective meaningful things, then we realize that the \( n \)-bein field is not yet fully determined by these settings. Yet metric and teleparallelism remain intact, if we substitute the \( n \)-beins of all points of the continuum with such \( n \)-beins that were derived out of the original ones by the rotation stated above. We denote this substitutability of the \( n \)-bein field as rotational invariance and establish: Only those mathematical relations that are rotational invariant can claim a real meaning.

Thus by keeping the coordinate system fixed, and a given metric and parallel connection, the \( h^a_\mu \) are not yet fully determined; there is a possible substitution which corresponds to the rotation invariance

\[ A^*_a = d_a m A_m, \ldots \]  

(6)

whereby \( d_a m \) is chosen orthogonal and independent of the coordinates. \( (A_a) \) is an arbitrary vector with respect to the local system, \( (A^*_a) \) the same vector with respect to the rotated local system. According to (1a), and using (6), it follows

\[ h^*_\mu a A^\mu = d_{am} h_{\mu m} A^\mu \]

or

\[ h^*_\mu a = d_{am} h_{\mu m}, \ldots \]  

(6a)

whereby

\[ d_{am} d_{bm} = d_{ma} d_{mb} = \delta_{ab}, \ldots \]  

(6b)

\[ \frac{\partial d_{am}}{\partial x^\nu} = 0, \ldots \]  

(6c)

Now the postulate of rotation invariance tells us that among the relations in which the quantities \( h \) appear, only those may be seen as meaningful, which are transformed into \( h^* \) of equal form, if \( h^* \) is introduced by eqns. (6). In other words: \( n \)-bein fields which are related by locally equal rotations are equivalent.
The rule of infinitesimal parallel transport of a vector from point \((x^\nu)\) to a neighboring point \((x^\nu + dx^\nu)\) is obviously characterized by

\[
d A_\alpha = 0 \ldots, \tag{7}\]

that means by the equation

\[
0 = d(h_{\mu \alpha} A^\mu) = \partial h_{\mu \alpha} A^\mu dx^\sigma + h_{\mu \alpha} dA^\mu = 0 \ldots
\]

Multiply by \(h^\nu_\alpha\) this equation becomes considering (5)

\[
dA^\nu = -\Delta^\nu_{\mu \sigma} A^\mu dx^\sigma \quad \tag{7a}
\]

with

\[
\Delta^\nu_{\mu \sigma} = h^\nu_\alpha \partial h_{\mu \alpha} \partial x^\sigma. \]

This law of parallel transport is rotation invariant and not symmetric with respect to the lower indices of the quantities \(\Delta^\nu_{\mu \sigma}\). If one transports the vector \((A)\) now according to this law along a closed path, the vector remains unaltered; this means, that the Riemannian tensor

\[
R^i_{k,l,m} = -\partial \Delta^i_{kl} \partial x^m + \Delta^i_{k,m} - \Delta^i_{l,m} \Delta^i_{k,l}
\]

built from the connection coefficients vanishes according to (7a), which can be verified easily. Besides this law of parallel transport there is that (nonintegrable) symmetric transport law due to the Riemannian metric (2) and (3). As is generally known, it is given by the equations

\[
\bar{d}A^\nu = -\Gamma^\nu_{\mu \tau} A^\mu dx^\tau \quad \tag{8}
\]

\[
\Gamma^\nu_{\mu \tau} = \frac{1}{2} g^{\nu \alpha} \left( \partial g_{\mu \alpha} \partial x^\tau + \partial g_{\tau \alpha} \partial x^\mu - \partial g_{\mu \tau} \partial x^\alpha \right).
\]

According to (3), the \(\Gamma^\nu_{\mu \tau}\) are expressed by the quantities \(h\) of the n-bein fields. Thereby one has to keep in mind that

\[
g^{\mu \nu} = h^\mu_\alpha h^\nu_\alpha \ldots \quad \tag{9}
\]

Because of this setting and due to (4) and (5) the equations

\[
g^{\mu \lambda} g_{\nu \lambda} = \delta^\mu_\nu \]

are fulfilled which define the \(g^{\mu \lambda}\) calculated from the \(g_{\mu \lambda}\). This transport law based on metric only is obviously rotation invariant in the above sense.

§ 3. Invariants and covariants

On the manifold we are considering, besides the tensors and invariants of RIEMANN-geometry which contain the quantities \(h\) only in the combination (3), other tensors and invariants exist, among which we will have a look at the simplest ones only.

If one starts with a vector \((A^\nu)\) in the point \(x^\nu\), with the shifts \(d\) and \(\bar{d}\), the two vectors

\[
A^\nu + dA^\nu
\]

and

\[
A^\nu + \bar{d}A^\nu
\]

are produced in the neighboring point \((x + dx^\nu)\). Thus the difference

\[
dA^\nu - \bar{d}A^\nu = (\Gamma^\nu_{\alpha \beta} - \Delta^\nu_{\alpha \beta}) A^\alpha dx^\beta
\]
has vector character as well. Therefore,
\[(\Gamma^\nu_{\alpha\beta} - \Delta^\nu_{\alpha\beta})\]
is a tensor, and also its skewsymmetric part
\[
\frac{1}{2}(\Delta^\nu_{\alpha\beta} - \Delta^\nu_{\beta\alpha}) = \Lambda^\nu_{\beta\alpha} \ldots
\]
The fundamental meaning of this tensor in the theory developed here results from the following: If this tensor vanishes, then the continuum is Euclidean. Namely, if
\[
0 = 2\Lambda^\nu_{\alpha\beta} = h^\nu_a \left( \frac{\partial h_{\alpha a}}{\partial x^\beta} - \frac{\partial h_{\beta a}}{\partial x^\alpha} \right),
\]
holds, then by multiplication with \(h_{\nu b}\) follows
\[
0 = \frac{\partial h_{\alpha b}}{\partial x^\beta} - \frac{\partial h_{\beta b}}{\partial x^\alpha}.
\]
However, one may assume
\[
h_{ab} = \frac{\partial \psi_b}{\partial x^a}.
\]
Therefore the field is derivable from \(n\) scalars \(\psi_b\). We now choose the coordinates according to the equation
\[
\psi_b = x^b
\]
Then, due to (7a) all the \(\Delta^\nu_{\beta\alpha}\) vanish, and the \(h_{\mu a}\) and the \(g_{\mu\nu}\) are constant.—
Since the tensor\(^4\) \(\Lambda^\nu_{\beta\alpha}\) is formally the simplest one admitted by our theory, this tensor shall be used as a starting point for characterizing such a continuum, and not the more complicated Riemannian curvature tensor. The most simple quantities which come in mind are the vector
\[
\Lambda^\alpha_{\mu\alpha}
\]
and the invariants
\[
g^{\mu\nu} \Lambda^\alpha_{\mu\beta} \Lambda^\beta_{\nu\alpha} \text{ and } g_{\mu\nu} g^{\alpha\sigma} g^{\beta\tau} \Lambda^\mu_{\alpha\beta} \Lambda^\nu_{\alpha\tau}
\]
From one of the latter ones (actually, from a linear combination of it), after multiplication with the invariant volume element
\[
h \, d\tau,
\]
(whereby \(h\) means the determinant \(|h_{\mu a}|\), \(d\tau\) the product \(dx_1...dx_n\), an invariant integral \(J\), may be built. The setting
\[
\delta J = 0
\]
then provides 16 differential equations for the 16 quantities \(h_{\mu a}\).
If laws with relevance to physics can be derived from this, shall be investigated later.— It clarifies things, to compare WEYL’S modification of the RIEMANNIAN theory to the one presented here:

WEYL: no comparison at a distance, neither of the absolute values, nor of directions of vectors.
RIEMANN: comparison at a distance for absolute values of vectors, but not of directions of vectors.
PRESENT THEORY: comparison of both absolute values and directions of vectors at a distance.\(^5\)

\(^4\)Tr. note: this is called torsion tensor in the literature.
\(^5\)Tr. note: This is the origin of the name distant parallelism as a synonym for absolute parallelism or teleparallelism, in German Fernparallelismus.
New Possibility for a Unified Field Theory of Gravitation and Electricity

Albert Einstein

translation by A. Unzicker and T. Case

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Some days ago I explained in a short note in these reports, how by using a $n$-bein field a geometric theory can be constructed that is based on the notion of a Riemann-metric and distant parallelism. I left open the question if this theory could serve for describing physical phenomena. In the meantime I discovered that this theory - at least in first approximation – yields the field equations of gravitation and electromagnetism in a very simple and natural manner. Thus it seems possible that this theory will substitute the theory of general relativity in its original form.

The introduction of this theory has as a consequence the existence of a straight line, that means a line of which all elements are parallel to each other. Naturally, such a line is not identical with a geodesic. Furthermore, contrarily to the actual theory of relativity, the notion of relative rest of two mass points exists (parallelism of two line elements that belong to two different world lines).

In order to apply the general theory in the implemented form to the field theory, one has to set the following conventions:

1. The dimension is 4 ($n = 4$).
2. The fourth local component $A_a$ (a=4) of a vector is purely imaginary, and so are the components of the 4th bein of the 4-bein, and also the quantities $h^\nu_4$ and $h_{\nu4}$. Of course, all the coefficients of $\gamma^{\mu\nu}$ ($= h_{\mu\alpha}h_{\nu\alpha}$) become then real. Thus, we choose the square of the modulus of a timelike vector to be negative.

§ 1. The assumed basic field law

For the variation of the field potentials $h_{\mu\alpha}$ (or $h^\alpha_\mu$) to vanish on the boundary of a domain the variation of the Hamiltonian integral should vanish:

$$\delta\{\int \mathcal{H}d\tau\} = 0,$$

with the quantities $h(= |h_{\mu\alpha}|)$, $g^{\mu\nu}$, $\Lambda_{\mu\beta}^\alpha \Lambda_{\nu\alpha}^\beta$, defined in the eqns. (9), (10) loc.it.\cite{footnote9}

\textit{tr. note:} such lines are nowadays called autoparallels.

\textit{tr. note:} 4-bein (tetrad, from German ‘bein’ = leg) has become a common expression in differential geometry.

\textit{Instead of this one could define the square of the length of the local vector $A_1^2 + A_2^2 + A_3^2 - A_4^2$ and introduce Lorentz-transformations instead of rotations of the local n-bein. In that case all the $h$ would become real, but one would loose the direct connection to the formulation of the general theory.}

\textit{tr. note:} this refers to the session report of June 7th, 1928.
The field $h$ should describe both the electrical and gravitational field. A ‘pure gravitational’ field means that in addition to eqn. (1) being satisfied the quantities

$$\phi_{\mu} = \Lambda_{\mu a}$$

vanish, which is a covariant and rotation invariant restriction.\(^{10}\)

§ 2. The field law in first approximation

If the manifold is the Minkowski-world of the theory of special relativity, one may choose the coordinate system in a way that

$$h_{11} = h_{22} = h_{33} = 1, h_{44} = j (= \sqrt{-1})$$

holds and the other $h_{\mu a}$ vanish. This system of values for $h_{\mu a}$ is a little inconvenient for calculations. Therefore in this paragraph for calculations we prefer to assume the $x_4$-coordinate to be purely imaginary; then, the Minkowski-world (absence of any field for a suitable choice of coordinates) can be described by

$$h_{\mu a} = \delta_{\mu a} \ldots$$

The case of infinitely weak fields can be described purposely by

$$h_{\mu a} = \delta_{\mu a} + k_{\mu a} \ldots$$

whereby the $k_{\mu a}$ are small values of first order. While neglecting quantities of third and higher order one has to replace (1a) with respect to (10) and (7a) loc. it. by

$$\mathcal{H} = \frac{1}{4} \left( \frac{\partial k_{\mu a}}{\partial x_{\beta}} - \frac{\partial k_{\beta a}}{\partial x_{\mu}} \right) \left( \frac{\partial k_{\alpha \beta}}{\partial x_{\mu}} - \frac{\partial k_{\alpha \mu}}{\partial x_{\beta}} \right).$$

(1b)

By performing the variation one obtains the field equations valid in first approximation

$$\frac{\partial^2 k_{\beta a}}{\partial x_{\mu}^2} - \frac{\partial^2 k_{\mu a}}{\partial x_{\mu} \partial x_{\beta}} + \frac{\partial^2 k_{\alpha \mu}}{\partial x_{\mu} \partial x_{\beta}} - \frac{\partial^2 k_{\beta \mu}}{\partial x_{\mu} \partial x_{\alpha}} = 0 \ldots$$

(5)

This are 16 equations\(^{11}\) for the 16 quantities $k_{\alpha \beta}$. Our task is now to see if this system of equations contains the known laws of gravitational and the electromagnetical field. For this purpose we introduce in (5) the $g_{\alpha \beta}$ and the $\phi_{\alpha}$ instead of the $k_{\alpha \beta}$. We have to define

$$g_{\alpha \beta} = h_{\alpha a} h_{\beta a} = \left( \delta_{\alpha a} + k_{\alpha a} \right) \left( \delta_{\beta a} + k_{\beta a} \right)$$

or in first order

$$g_{\alpha \beta} - \delta_{\alpha \beta} = g_{\alpha \beta} = k_{\alpha \beta} + k_{\beta \alpha} \ldots.$$  

(6)

From (2) one obtains further the quantities of first order, precisely

$$2 \phi_{\alpha} = \frac{\partial k_{\alpha \mu}}{\partial x_{\mu}} - \frac{\partial k_{\mu a}}{\partial x_{\alpha}} \ldots$$

(2a)

By exchanging $\alpha$ and $\beta$ in (5) and adding the thus obtained structuring of (5) at first one gets

$$\frac{\partial^2 g_{\alpha \beta}}{\partial x_{\mu}^2} - \frac{\partial^2 k_{\alpha \mu}}{\partial x_{\mu} \partial x_{\beta}} - \frac{\partial^2 k_{\mu \beta}}{\partial x_{\mu} \partial x_{\alpha}} = 0.$$  

\(^{10}\)There is still a certain ambiguity in interpreting, because one could characterize the gravitational field by the vanishing of $\frac{\partial \phi_{\mu}}{\partial x_{\nu}} - \frac{\partial \phi_{\nu}}{\partial x_{\mu}}$ as well.

\(^{11}\)Naturally, between the field equations there exist four identities due to the general covariance. In the first approximation treated here this is expressed by the fact that the divergence taken with respect to the index $a$ of the l.h.s. of (5) vanishes identically.
If to this equation the two equations

\[-\frac{\partial^2 k_{\alpha \mu}}{\partial x_\mu \partial x_\beta} + \frac{\partial^2 k_{\mu \mu}}{\partial x_\alpha \partial x_\beta} = -2 \frac{\partial \phi_\alpha}{\partial x_\beta},\]

\[-\frac{\partial^2 k_{\beta \mu}}{\partial x_\mu \partial x_\alpha} + \frac{\partial^2 k_{\mu \mu}}{\partial x_\alpha \partial x_\beta} = -2 \frac{\partial \phi_\beta}{\partial x_\alpha},\]

are added, following from (2a), one obtains according to (6)

\[\frac{1}{2}\left(-\frac{\partial^2 g_{\alpha \beta}}{\partial x_\mu^2} + \frac{\partial^2 g_{\mu \alpha}}{\partial x_\mu \partial x_\beta} + \frac{\partial^2 g_{\mu \beta}}{\partial x_\mu \partial x_\alpha} - \frac{\partial^2 g_{\mu \mu}}{\partial x_\alpha \partial x_\beta} + \right) = \frac{\partial \phi_\alpha}{\partial x_\beta} + \frac{\partial \phi_\beta}{\partial x_\alpha} \ldots \quad (7)\]

The case of the absence of an electromagnetic field is characterized by the vanishing of \(\phi_\mu\). In this case (7) is in first order equivalent to the equation

\[R_{\alpha \beta} = 0\]

used as yet in the theory of general relativity (\(R_{\alpha \beta} = \) contracted Riemann tensor). With the help of this it is proved that our new theory yields the law of a pure gravitational field in first approximation correctly.

By differentiation of (2a) by \(x_\alpha\), one gets the equation

\[\frac{\partial \phi_\alpha}{\partial x_\alpha} = 0. \quad (8)\]

according to (5) and contraction over \(\alpha\) and \(\beta\). Taking into account that the l.h.s. \(L_{\alpha \beta}\) of (7) fulfills the identity

\[\frac{\partial}{\partial x_\beta} (L_{\alpha \beta} - \frac{1}{2} \delta_{\alpha \beta} L_{\sigma \sigma}) = 0,\]

from (7) follows

\[\frac{\partial^2 \phi_\alpha}{\partial^2 x_\beta} + \frac{\partial^2 \phi_\beta}{\partial x_\alpha \partial x_\beta} - \frac{\partial}{\partial x_\alpha} \left(\frac{\partial \phi_\sigma}{\partial x_\beta}\right) = 0\]

or

\[\frac{\partial^2 \phi_\alpha}{\partial^2 x_\beta} = 0. \ldots \quad (9)\]

The equations (8) and (9) are, as it is well known, equivalent to Maxwell’s equations for empty space. The new theory thus also yields Maxwell’s equations in first approximation.

According to this theory, the separation of the gravitational and electromagnetic field seems arbitrary however. Furthermore, it is clear that the eqns. (5) state more than the eqns. (7), (8) and (9) together. After all it is remarkable that the electric field does not enter the field equations quadratically.

**Note added in proof.** One obtains very similar results by starting with the Hamilton function

\[\mathcal{H} = h \ g_{\mu \nu} g^{\sigma \tau} \Lambda_{\alpha \beta}^\mu \Lambda_{\sigma \tau}^\nu.\]

Thus for the time being there remains a certain insecurity regarding the choice of \(\mathcal{H}\).
In the present work I would like to describe a theory I have been working on for a year; it will be exposed in a manner that it can be understood comfortably by everyone who is familiar with the theory of general relativity. The following exposure is necessary, because due to coherences and improvements found in the meantime reading the earlier work would be a useless loss of time. The topic is presented in a way that seems most serviceable for comfortable access. I learned, especially with the help of Mr. Weitzenböck and Mr. Cartan, that the dealing with the continua we are talking about is not new. Mr. Cartan kindly wrote an essay about the history of the relevant mathematical topic in order to complete my paper; it is printed right after this paper in the same review. I would also like to thank Mr. Cartan heartily at this point for his valuable contribution. The most important and undisputable new result of the present work is the finding of the most simple field laws that can be applied to a Riemannian manifold with distant parallelism. I am only going to discuss their physical meaning briefly.

§ 1. The structure of the continuum

Since the number of dimensions has no impact on the following considerations, we suppose a \( n \)-dimensional continuum. To take into account the facts of metrics and gravitation we assume the existence of a Riemann-metric. In nature there also exist electromagnetic fields, which cannot be described by Riemannian metrics. This arouses the question: How can we complement our Riemannian spaces in a natural, logical way with an additional structure, so that the whole thing has a uniform character?

The continuum is (pseudo-)Euclidean in the vicinity of every point \( P \). In every point there exists a local coordinate system of geodesics (i.e. an orthogonal \( n \)-bein), in relation to which the theorem of Pythagoras is valid. The orientation of these \( n \)-beins is not important in a Riemannian manifold. We would now like to assume that these elementary Euclidean spaces are governed by still another direction law. We are also going to assume, that it makes sense to speak of a parallel orientation of all \( n \)-beins together, applying this to space structure like in Euclidean geometry (which would be senseless in a space with metrical structure only).

In the following we are going to think of the orthogonal \( n \)-beins as being always in parallel orientation. The in its self arbitrary orientation of the local \( n \)-bein in one point \( P \) then determines the orientation of the local \( n \)-beins in all points of the continuum uniquely. Our task now is to set up the most simple restrictive laws which can be applied to such a continuum. Doing so, we hope to derive the general laws of nature, as the previous theory of general relativity tried this for gravitation by applying a purely metrical space structure.
§ 2. Mathematical description of the space structure

The local $n$-bein consists of $n$ orthogonal unit vectors with components $h_s^\nu$ with respect to any Gaussian coordinate system. Here as always a lower Latin index indicates the affiliation to a certain bein of the $n$-beins, a Greek index - due to its upper or lower position - the covariant or contravariant transformation character of the relevant entity with respect to a change of the Gaussian coordinate system. The general transformation property of the $h_s^\nu$ is the following. If all local systems or $n$-beins are twisted in the same manner, which is a correct operation of course, and a new Gaussian coordinate system is introduced at the same time, the following transformation law then exists in-between the new and old $h_s^\nu$

$$h_s^\nu' = \alpha_{st} \frac{\partial x^t'}{\partial x^s} h_t^\alpha,$$  

whereas the constant coefficients $\alpha_{st}$ form an orthogonal system:

$$\alpha_{sa}\alpha_{sb} = \delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b \end{cases}$$  

Without problems the transformation law (1) can be generalized onto objects which components bear an arbitrary number of local indices and coordinate indices. We call such objects tensors. Out of this the algebraic laws of tensors (addition, multiplication, contraction by Latin and Greek indices) follow immediately.

We call $h_s^\nu$ the components of the fundamental tensor. If a vector in the local system has components $A_s$, and the coordinates $A^\nu$ with respect to the Gaussian system, it follows out of the meaning of the $h_s^\nu$:

$$A^\nu = h_s^\nu A_s$$  

or – resolved with respect to the $A_s$–

$$A_s = h_{su} A^\nu$$  

The tensorial character of the normalized subdeterminants $h_{su}$ of the $h_s^\nu$ follows out of (4). $h_{su}$ are the covariant components of the fundamental tensor. Between $h_{su}$ and $h_s^\nu$ there are the relations

$$h_{su} h_s^\nu = \delta_{\nu}^\mu = \begin{cases} 1, & \text{if } \mu = \nu \\ 0, & \text{if } \mu \neq \nu \end{cases}$$

$$h_{su} h_t^\mu = \delta_{st}$$  

Due to the orthogonality of the local system we obtain the absolute value of the vector

$$A^2 = A_s^2 = h_{su} h_{sv} A^\mu A^\nu = g_{\mu\nu} A^\mu A^\nu;$$  

Therefore,

$$g_{\mu\nu} = h_{su} h_{sv}$$  

are the coefficients of the metric.

The fundamental tensor allows (cfr. (3) and (4)) to transform local indices into coordinate indices and vice versa (by multiplication and contraction), so that it comes down to pure convention, with which type of tensors one likes to operate.

Obviously the following relations hold:

$$A_\nu = h_{su} A_s,$$  

(3a)

$$A_s = h_s^\nu A_\nu,$$  

(4a)
Furthermore, we have the relation of determinants

\[ g = |g_{\sigma\tau}| = |h_{\alpha\sigma}|^2 = h^2 \]  

(8)

Therefore, the invariant of the volume element \( \sqrt{g} d\tau \) takes the form \( h d\tau \). To take into account the particular properties of time, it is most comfortable to set the \( x^4 \)-coordinate (both local and general) of our 4-dimensional space-time continuum purely imaginary and also all tensor components with an odd number of indices 4.

§ 3. Differential relations

Now we denote \( \delta \) the change of the components of a vector or tensor during a 'parallel displacement' in the sense of Levi-Civita during the transition to a infinitely neighboring point of the continuum; now it follows out of the above

\[ 0 = \delta A_s = \delta(h_{s\alpha} A^\alpha) = \delta(h_s^\alpha A_\alpha) \]  

(9)

Resolving the brackets yields

\[
\begin{align*}
 h_{s\alpha} \delta A^\alpha + A^\alpha h_{s\alpha,\beta} \delta x^\beta &= 0, \\
 h_s^\alpha \delta A_\alpha + A_\alpha h_s^{\alpha,\beta} \delta x^\beta &= 0,
\end{align*}
\]

whereas the colon indicates ordinary differentiation by \( x^\beta \). Resolving of the equation yields

\[
\begin{align*}
 \delta A^\sigma &= -A^\alpha \Delta_\alpha^{\sigma,\beta} \delta x^\beta, \\
 \delta A_\sigma &= A_\alpha \Delta^{\sigma,\beta}_\alpha \delta x^\beta,
\end{align*}
\]

(10)  

(11)

whereby we set

\[
\Delta^{\sigma,\beta}_\alpha = h_s^{\sigma} h_{s\alpha,\beta} = -h_{s\alpha} h_{s,\beta}^{\sigma} \]  

(12)

(The last conversion is based on (5)).

This law of parallel displacement is – contrarily to Riemannian geometry – in general not symmetric. If it is, we have Euclidean geometry, because

\[
\Delta_\alpha^{\sigma,\beta} - \Delta^{\sigma,\beta}_\alpha = 0
\]

or

\[
h_{s\alpha,\beta} - h_{s\beta,\alpha} = 0.
\]

But then

\[
h_{sa} = \frac{\partial \psi_s}{\partial x_\alpha}
\]

holds. If one chooses the \( \psi_s \) as new variables \( x'_s \), we obtain

\[ \text{tr. note: the connection } \Delta \text{ is nowadays usually denoted as } \Gamma. \text{ Cfr. Schouten, Ricci Calculus (Springer, 1954), chap. III (1.2)} \]
\[ h_{\sigma\alpha} = \delta_{\sigma\alpha}, \quad (13) \]

proving the statement.

**Covariant differentiation.** The local components of a vector are invariant with respect to any coordinate transformation. Out of this follows immediately the tensorial character of the differential quotient

\[ A_{s,\alpha}. \quad (14) \]

Because of (4a) this can be replaced by

\[ (h_s^{\sigma} A_\sigma)_{,\alpha}, \]

and the tensorial character of

\[ h_s^{\sigma} A_{\sigma,\alpha} + A_{\sigma} h_s^{\sigma}_{,\alpha}, \]

follows. Equally (after multiplication with \( h_{s\tau} \)) the tensorial character of

\[ A_{\tau,\alpha} + A_{\sigma} h_s^{\sigma}_{,\alpha} h_{s\tau}, \]

and of

\[ A_{\tau,\alpha} - A_{\sigma} h_s^{\sigma} h_{s\tau,\alpha} \]

and (see (16)) of

\[ A_{\tau,\alpha} - A_{\sigma} \Delta_{\alpha}^{\sigma}_{\tau}. 13 \]

We call this covariant derivative \((A_{\tau;\alpha})\) of \(A_{\tau}\).

Therefore, we obtain the law of covariant differentiation

\[ A_{\sigma;\tau} = A_{\sigma,\tau} - A_{\alpha} \Delta_{\alpha}^{\sigma}_{\tau} \quad (15) \]

Analogously, out of (3) follows the formula

\[ A_{\sigma;\tau} = A_{\sigma,\tau} + A_{\alpha} \Delta_{\alpha}^{\sigma}_{\tau}. \quad (16) \]

The result is the law of covariant differentiation for arbitrary tensors. We illustrate this giving an example:

\[ A_{a}^{\sigma}_{\tau;\rho} = A_{a}^{\sigma}_{\tau,\rho} + A_{a}^{\alpha}_{\tau} \Delta_{\alpha}^{\sigma}_{\rho} - A_{a}^{\sigma}_{\alpha} \Delta_{\alpha}^{\sigma}_{\tau} \rho. \quad (17) \]

By means of the fundamental tensor \( h_s^{\alpha} \) we are allowed to transform local (Latin) indices in coordinate (Greek) indices, so we are free to favor the local or coordinate indices when formulating some tensor relations. The first approach is preferred by the Italian colleagues (Levi-Civita, Palatini), while I have preferably used coordinate indices.

**Divergence.** By contraction of the covariant differential quotient one obtains the divergence as in the absolute differential calculus based on metrics only. E.g., one gets the tensor

\[ A_{\alpha\tau} = A_{\alpha}^{\sigma}_{\tau;\sigma}. \]
out of (21) by contraction of the indices \( \sigma \) and \( \rho \).
In earlier papers I even introduced other divergence operators, but I do not accredit special significance to those any more.

**Covariant differential quotients of the fundamental tensor.**
One can easily find out of the formulas derived above, that the covariant derivatives and divergences of the fundamental tensor vanish. E.g. we have

\[
h_{s}^{\nu,\tau} \equiv h_{s}^{\nu,\tau} + h_{s}^{\alpha} \Delta_{\alpha}^{\nu,\tau} \equiv \delta_{st}(h_{t}^{\nu,\tau} + h_{t}^{\alpha} \Delta_{\alpha}^{\nu,\tau})
\]

\[
\equiv h_{s}^{\alpha}(h_{t\alpha} h_{t}^{\nu,\tau} + \Delta_{\alpha}^{\nu,\tau}) \equiv h_{s}^{\alpha}(\gamma_{\alpha}^{\nu,\tau} + \Delta_{\alpha}^{\nu,\tau}) \equiv 0.
\]

Analogously we can prove

\[
h_{s}^{\nu,\tau} \equiv g^{\mu\nu,\tau} \equiv g_{\mu\nu,\tau} \equiv 0. \quad (18a)
\]

Likewise, the divergences \( h_{s}^{\nu,\nu} \) and \( g^{\mu\nu,\nu} \) obviously vanish.

**Differentiation of tensor products.** As it is apparent in the well-known differential calculus the covariant differential quotient of a tensor product can be expressed by the differential quotient of the factors. If \( S \cdot \) and \( T \cdot \) are tensors of arbitrary index character,

\[
(S \cdot T \cdot)_{\alpha} = S_{\cdot,\alpha} T \cdot + T \cdot,\alpha S \cdot.
\]

follows. Out of this and out of the vanishing covariant differential quotient of the fundamental tensor it follows, that the latter may be interchanged with the differentiation symbol (\( \cdot \)).

**"Curvature".** Out of the hypothesis of "distant parallelism" and out of equation (9) we obtain the integrability of the displacement law (10) and (11). Out of this follows

\[
0 \equiv -\Delta^{\tau}_{\kappa \lambda, \mu} \equiv -\Delta^{\tau}_{\kappa \lambda, \mu} + \Delta^{\tau}_{\kappa \mu, \lambda} + \Delta^{\sigma}_{\sigma \lambda} \Delta^{\mu}_{\kappa \mu} - \Delta^{\sigma}_{\sigma \mu} \Delta^{\mu}_{\kappa \lambda}.
\]

In order to be expressed by the entities \( h \), the \( \Delta \)'s must comply to these conditions (cfr. (12)). Looking at (20), it is clear that the characteristic laws of the manifold in consideration here must be very different from the earlier theory. Though according to the new theory all tensors of the earlier theory exist, in particular the Riemannian curvature tensor calculated from the Christoffel symbols. But according to the new theory there are simpler and more elementary tensorial objects, that can be used for formulating the field laws.

**The tensor** \( \Lambda \). If we differentiate a scalar \( \psi \) twice covariantly, we obtain according to (15) the tensor

\[
\phi_{\sigma,\tau} - \phi_{,\alpha} \Delta^{\alpha}_{\sigma,\tau}.
\]

From this follows at once the tensorial character of

\[
\frac{\partial \phi}{\partial x_{\alpha}}(\Delta^{\alpha}_{\sigma,\tau} - \Delta^{\alpha}_{\tau,\sigma}).
\]

Interchanging \( \sigma \) and \( \tau \) a new tensor emerges and the subtraction yields the tensor

\[
\Lambda^{\alpha}_{\sigma,\tau} = \Delta^{\alpha}_{\sigma,\tau} - \Delta^{\alpha}_{\tau,\sigma}. \quad (21)
\]

According to this theory there is a tensor containing the components \( h_{\sigma\alpha} \) of the fundamental tensor and its first differential quotients only. We already proved that a vanishing fundamental tensor causes the validity of Euclidean geometry (cfr. (13)). Therefore, a natural law for such a continuum will

\[\text{tr. note: cfr. Schouten III, (2.13)}\]

\[\text{Cartans torsion tensor is nowadays usually denoted as } T \text{ or } S \text{ (Schouten)}\]
consist of conditions for this tensor.
By contraction of the tensor Λ we obtain

\[ \phi_\sigma = \Lambda_\sigma^\alpha \phi_\alpha. \]  \(16\)

A vector which, as I suspected earlier, could take the part of the electromagnetic potential in the present theory, but ultimately I do not uphold this view.

**Changing rule of differentiation.** If a tensor \( T \) is differentiated twice covariantly, the important rule holds

\[ T_{;\sigma;\tau} - T_{;\tau;\sigma} \equiv -T_{;\alpha} \Lambda_\sigma^\alpha. \]  \(23\)

**Proof.** If \( T \) is a scalar (tensor without Greek index), we obtain the proof without effort using (15).
In this special case we will find the proof of the general theorem.
The first remark we would like to make is, that according to the theory discussed here parallel vector fields do exist. These vector fields have the same components in all local systems. If \( a_\alpha \) is such a vector field, it satisfies the condition

\[ a_\alpha;_\sigma = 0 \quad \text{or} \quad a_\alpha;_\sigma = 0 \]

which can be proven easily.
Using such parallel vector fields the changing rule easily leads back to the rule for a scalar. For the sake of simplicity, we perform the proof for a tensor \( T^\lambda \) with only one index. If \( \phi \) is a scalar, the first thing that follows out of the definitions (16) and (21) is

\[ \phi_{;\sigma;\tau} - \phi_{;\tau;\sigma} \equiv -\phi_{;\alpha} \Lambda_\sigma^\alpha. \]

If we put the scalar \( a_\lambda T^\lambda \) into this equation for \( \phi \), \( a_\lambda \) being a parallel vector field, \( a_\lambda \) may be interchanged with the differentiation symbol at every covariant differentiation, therefore \( a_\lambda \) appears as a factor in all the terms. Therefore, one obtains

\[ [T^\lambda_{;\sigma;\tau} - T^\lambda_{;\tau;\sigma} + T^\lambda_{;\alpha} \Lambda_\sigma^\alpha]a_\lambda = 0. \]

This identity must hold for any choice of \( a_\lambda \) in a certain position, therefore the bracket vanishes, and we have finished our proof. The generalization for tensors with any number of Greek indices is obvious.

**Identities for the tensor \( \Lambda \).** Permuting the indices \( \kappa, \lambda, \mu \) in (20), adding the three identities, and by appropriate summing-up of the terms with respect to (21) one obtains

\[ 0 \equiv (\Lambda_{\kappa,\lambda;\mu} + \Lambda_{\lambda,\mu;\kappa} + \Lambda_{\mu,\kappa;\lambda}) + \Delta_{\sigma,\kappa} \Lambda_{\mu,\sigma} + \Delta_{\sigma,\lambda} \Lambda_{\mu,\sigma} + \Delta_{\sigma,\mu} \Lambda_{\sigma,\lambda}. \]  \(17\)

We convert this identity by introducing covariant instead of ordinary derivatives of the tensors \( a_\lambda \) (see (17)); so we acquire the identity

\[ 0 \equiv (\Lambda_{\kappa,\lambda;\mu} + \Lambda_{\lambda,\mu;\kappa} + \Lambda_{\mu,\kappa;\lambda}) + (\Lambda_{\kappa,\alpha} \Lambda_{\alpha,\mu} + \Lambda_{\lambda,\alpha} \Lambda_{\mu,\kappa} + \Lambda_{\mu,\alpha} \Lambda_{\kappa,\lambda}). \]  \(24\)

In order to express the \( a_\lambda \)'s by the \( h \) in the above manner, this condition must be satisfied.
Contraction of the above equation by the indices \( \tau \) and \( \mu \) yields the identity

\[ 0 \equiv \Lambda_{\kappa,\lambda;\alpha} + \phi_{\lambda,\kappa} \equiv -\phi_{\alpha} \Lambda_{\kappa,\lambda}. \]

or

\[ \Lambda_{\kappa,\lambda;\alpha} + \phi_{\lambda,\kappa} \phi_{\kappa,\lambda} \]  \(18\)

where \( \phi_{\lambda} \) stands for \( \Lambda_{\lambda;\alpha} \) (22).
§ 4. The field equations

The most simple field equations we desired to find will be conditions for the tensor \( \Lambda^{\alpha}_{\mu \nu} \). The number of \( h \)-components is \( n^2 \), of which \( n \) remain indeterminate due to general covariance; therefore the number of independent field equations must be \( n^2 - n \). On the other hand, the higher number of possibilities a theory cuts down on (without contradicting experience), the more satisfactory it is. Therefore, the number \( Z \) of field equations should be as large as possible. If \( \bar{Z} \) denotes the number of identities in-between the field equations, \( Z - \bar{Z} \) must be equal to \( n^2 - n \).

According to the change rule of differentiation:

\[
\Lambda^{\alpha}_{\mu \nu; \alpha} - \Lambda^{\alpha}_{\mu \nu; \alpha} = \Lambda^{\sigma}_{\mu \tau; \alpha} \Lambda^{\alpha}_{\sigma \tau} \equiv 0.19 
\]

holds. An underlined index indicates "pulling up" and "pulling down" of an index, respectively, e.g.

\[
\Lambda^{\alpha}_{\mu \nu} \equiv \Lambda^{\alpha}_{\beta \gamma} g^{\mu \beta} g^{\nu \gamma}, \quad \Lambda^{\alpha}_{\mu \nu} \equiv \Lambda^{\beta}_{\mu \nu} g_{\alpha \beta}.
\]

We write this Identity (26) in the form

\[
G^{\mu \alpha} - F^{\mu \nu} + \Lambda^{\sigma}_{\mu \tau} F^{\sigma \tau} \equiv 0, \quad (26a)
\]

with the following settings

\[
G^{\mu \alpha} \equiv \Lambda^{\alpha}_{\mu \nu} - \Lambda^{\sigma}_{\mu \tau} \Lambda^{\alpha}_{\sigma \tau}, \quad (27)
\]

\[
F^{\mu \nu} \equiv \Lambda^{\alpha}_{\mu \nu}, \quad (28)
\]

Now we make an ansatz for the field equations:

\[
G^{\mu \alpha} = 0, \quad (29)
\]

\[
F^{\mu \alpha} = 0. \quad (30)
\]

These equations seem to contain an forbidden overdetermination, because their number is \( n^2 + n(n-1)/2 \), while at first hand it is only known to satisfy the identities (26a).

Linking (25) with (30) it follows, that the \( \phi_k \) can be derived from a potential. Therefore, we set

\[
F^{\kappa} = \phi^{\kappa} - \frac{\partial \log \psi}{\partial x^{\kappa}} = 0. \quad (31)
\]

(31) is completely equivalent with (30). The equations (29), (31) combined are \( n^2 + n \) equations for \( n^2 + 1 \) functions \( h_{\alpha \nu} \) and \( \psi \). Besides (26a) there is, however, another system of identities between these equations we will derive now.

If \( G^{\mu \alpha} \) denotes the antisymmetric part of \( G^{\mu \alpha} \), one can figure out directly from (27)

\[
2 G^{\mu \alpha} = S^{\mu \alpha}_{\mu \nu} + \frac{1}{2} S^{\mu \alpha}_{\mu \tau} \Lambda^{\alpha}_{\sigma \tau} - \frac{1}{2} S^{\alpha \mu}_{\sigma \tau} \Lambda^{\alpha}_{\sigma \tau} + F^{\mu \alpha}, \quad (32)
\]

For the sake of abbreviation we introduce the totally skew-symmetric tensor

\[\text{tr. note: cfr. Schouten III, (4.9)}\]
\[ S_{\mu \nu}^\alpha = \Lambda_{\mu \nu}^\alpha + \Lambda_{\nu \mu}^\nu + \Lambda_{\mu \nu}^\nu. \]  
(33)

Figuring out the first term of (32) yields
\[ 2 \mathcal{G}^{\mu \alpha} = S_{\mu \alpha}^\nu - S_{\mu \nu}^\sigma \Lambda_{\nu \sigma}^\nu + F^{\mu \alpha}, \]  
(34)

But now, with respect to the definition of \( F_k \) (31)
\[ \Delta_{\sigma \nu}^\nu - \Delta_{\nu \sigma}^\nu \equiv \Lambda_{\sigma \nu}^\nu \equiv \phi_{\sigma} \equiv F_{\sigma} \equiv \frac{\partial \log \psi}{\partial x^{\sigma}}, \]  
(35)
or
\[ \Delta_{\sigma \nu}^\nu = \frac{\partial \log \psi}{\partial x^{\sigma}} + F_{\sigma} \]  
(35)
holds. Therefore, (34) takes the form
\[ h\psi (2 \mathcal{G}^{\mu \alpha} - F_{\mu \alpha} + S_{\mu \alpha}^\sigma F_{\sigma}) \equiv \frac{\partial}{\partial x^\sigma} (h \psi S_{\mu \alpha}^\sigma) \]  
(34b)

Due to the antisymmetry the desired system of identical equations follows
\[ \frac{\partial}{\partial x^\alpha} [h \psi (2 \mathcal{G}^{\mu \alpha} - F_{\mu \alpha} + S_{\mu \alpha}^\sigma F_{\sigma})] \equiv 0 \]  
(36)

These are at first \( n \) identities, but only \( n - 1 \) of them are linear independent from each other. Because of the antisymmetry \[ \{ \} \equiv 0 \] holds independently no matter what one inserts in \( G^{\mu \alpha} \) and \( F_{\mu} \).

In the identities (4) and (36) you have to think of \( F^{\mu \alpha} \) being expressed by \( F_{\mu} \) according to the following relation which was derived from (31)
\[ F_{\mu \alpha} \equiv F_{\mu , \alpha} - F_{\alpha , \mu}. \]  
(31a)

Now we are able to prove the compatibility of the field equations (29), (30) or (29), (31), respectively.
First of all we have to show that the number of field equations minus the number of (independent) identities is smaller by \( n \) than the number of field variables. We have
\[
\begin{array}{c}
\text{number of field equations (29) (31) :} \\
\text{number of (independent) identities:} \\
\text{number of field variables:}
\end{array}
\begin{array}{c}
\quad n^2 + n \\
\quad n + n - 1 \\
\quad n^2 + 1, \\
\quad (n^2 + n) - (n + n - 1) = (n^2 + 1) - n
\end{array}
\]

As we see the number of identities just fits. We do not stop here, but prove the following

**Proposition.** If in a cross section \( x^n = \text{const.} \) all differential equations are satisfied and, in addition, \( (n^2 + 1) - n \) of them are properly chosen everywhere, then all \( n^2 + n \) equations are fulfilled anywhere.

**Proof.** If all equations are fulfilled in the cross section \( x^n = a \) and if these equations, that correspond to setting to zero the below, are fulfilled everywhere, we obtain:
\[
\begin{array}{cccc}
F_1 & \ldots & F_{n-1} & F_n \\
G^{1 \ 1} & \ldots & G^{1 \ n-1} & \\
& \ldots & \\
G^{n-1 \ 1} & \ldots & G^{n-1 \ n-1}.
\end{array}
\]

Then from (4) follows, that the \( F^{\mu \alpha} \) vanish everywhere. Now one deduces from (36), that in an neighboring cross section \( x^n = a + da \) the skew-symmetric \( \mathcal{G}^{\mu \alpha} \) for \( \alpha = n \) must vanish as well \(^{20}\). Out of (26a) it then follows, that in addition the symmetric \( \mathcal{G}^{\mu \alpha} \) for \( \alpha = n \) at the adjacent cross section \( x^n = a + da \) must vanish. Repeating this kind of deduction proves the proposition.

\(^{20}\)The \( \frac{\partial \mathcal{G}^{\mu \alpha}}{\partial x^n} \) vanish for \( x^n = a \).
§ 5. First approximation

We are now going to deal with a field that shows very little difference from an Euclidean one with ordinary parallelism. Then we may set

$$h_{s\nu} = \delta_{s\nu} + \bar{h}_{s\nu}, \quad (37)$$

where $\bar{h}_{s\nu}$ is infinitely small at first order, higher order terms are neglected. Then, according to (5) and (6), we have to set

$$h_{s'\nu} = \delta_{s\nu} - \bar{h}_{\nu s}. \quad (38)$$

In first approximation, the field equations (29), (31) read

$$\bar{h}_{a\mu, \nu, \nu} - \bar{h}_{a\nu, \nu, \mu} = 0, \quad (39)$$

$$\bar{h}_{a\mu, a, \nu} - \bar{h}_{a\nu, a, \mu} = 0. \quad (40)$$

we substitute equation (31) by

$$\bar{h}_{a\nu, a} = \chi_{\nu}. \quad (40a)$$

We claim now that there is an infinitesimal coordinate transformation $x'^{\nu} = x^{\nu} - \xi_{\nu}$, which causes all the variables $\bar{h}_{a\nu, \nu}$ and $\bar{h}_{a\nu, a}$ to vanish.

Proof. First we prove that

$$\bar{h}_{\mu\nu}' = \bar{h}_{\mu\nu} \xi_{\nu}, \quad (41)$$

Therefore,

$$\bar{h}_{a\nu, \nu}' = \bar{h}_{a\nu, \nu} + \xi_{\nu}, \quad (42)$$

$$\bar{h}_{a\nu, a}' = \bar{h}_{a\nu, a} + \xi_{a}, \quad (43)$$

The right sides vanish because of (40a), if the following equations are fulfilled

$$\xi_{\nu}, \nu = -\bar{h}_{a\nu, \nu}, \quad (42)$$

$$\xi_{a} = -\chi. \quad (43)$$

But these $n + 1$ equations for $n$ variables $\xi_{\alpha}$ are compatible, because of (40a)

$$(-\bar{h}_{a\nu, \nu}), a - (-\chi), \nu, \nu = 0.$$

Choosing new coordinates, the field equations read

$$\bar{h}_{a\mu, \nu\nu} = 0, \quad (43)$$

$$\bar{h}_{a\mu, a} = 0,$$

$$\bar{h}_{a\mu, \mu} = 0,$$
If we now separate $h_{\alpha\nu}$ according to the equations

\[
\begin{align*}
\bar{h}_{\alpha\mu} + \bar{h}_{\mu\alpha} &= \bar{g}_{\alpha\mu}, \\
\bar{h}_{\alpha\mu} - \bar{h}_{\mu\alpha} &= a_{\alpha\mu},
\end{align*}
\]

where $\delta_{\alpha\mu} + \bar{g}_{\alpha\mu}(= g_{\mu\nu})$ determines metrics in first approximation, thus the field equations take the simple form

\[
\begin{align*}
\bar{g}_{\alpha\mu, \sigma, \sigma} &= 0, \quad (44) \\
\bar{g}_{\alpha\mu, \mu} &= 0, \quad (45) \\
a_{\alpha\mu, \sigma, \sigma} &= 0, \quad (46) \\
a_{\alpha\mu, \mu} &= 0. \quad (47)
\end{align*}
\]

One is led to suppose, that $\bar{g}_{\alpha\nu}$ and $a_{\alpha\mu}$ represent the gravitational and the electromagnetic field in first approximation respectively. (44), (45) correspond to Poisson’s equation, (46), (47) to Maxwell’s equations of the empty space. It is interesting that the field laws of gravitation seem to be separated from those of the electromagnetic field, a fact which is in agreement with the observed independence of the two fields. But in a strict sense none of them exists separately.

Regarding the covariance of the equations (44) to (47) we note the following. For the $h_{s\mu}$’s generally the transformation law

\[
h'_{s\mu} = \alpha_{st} \frac{\partial x^\sigma}{\partial x'^t} h_{t\sigma}
\]

holds. If the coordinate transformation is chosen linear and orthogonal as well as conform with respect to the twist of the local systems, that is

\[
x'^{\mu'} = \alpha_{\mu\sigma} x^{\sigma}, \quad (48)
\]

we acquire the transformation law

\[
h'_{s\mu} = \alpha_{st} \alpha_{\mu\sigma} h_{t\sigma}, \quad (49)
\]

which is exactly the same as for tensors in special relativity. Because of (48) the same transformation law holds for the $\delta_{s\mu}$, so it also holds for the $\bar{h}_{s\mu}$, $\bar{g}_{s\mu}$, and $a_{s\mu}$. With respect to such transformations the equations (44) to (47) are covariant.

§ 6. Outlook

The big appeal of the theory exposed here, lies in its unifying structure and the high-level (but allowed) overdetermination of the field variables. I was able to show that the field equations yield equations, in first-order approximation, that correspond to the Newton-Poisson theory of gravitation and to Maxwell’s theory of the electromagnetic field. Nevertheless I’m still far away from claiming the physical validity of the equations I derived. The reason for that is, that I did not succeed in deriving equations of motion for particles yet.