Dynamical Voronoi tessellation

I. The two-dimensional case

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Summary. Given a configuration of points, a new procedure for constructing the corresponding Voronoi tessellation is derived. This procedure based on the local scanning of latticed squares of 400 x 400 points could be easily applied to explore intermediate positions in which the signals/expansions starting from the various injection points have not yet collided. The derived algorithm is then applied to the supernovae explosions in the galactic plane.

Key words: interstellar medium: bubbles – interstellar medium: radiation field – supernovae and supernova remnants: general

1. Introduction

Thanks to the increased computing speed and dynamical memory of the present computers a certain number of old problems could be better studied; a typical example is that of the tessellation first introduced by Dirichlet (1850) and Voronoi (1908).

The astrophysical applications of the Voronoi diagrams started with a pioneering work of Kiang (1966) where using 80 x 80 lattice points the size-distribution of random polygons was determined. The problem was then reconsidered 21 years later by Icke and Van de Weygaert (1987), where different distributions of nuclei were considered with a view towards applying the resulting “Voronoi foams” to the galaxy distribution in the universe. The previous authors also performed a statistical analysis of the circumference, the area, the number of vertices and the length of each wall for every cell.

Here, starting from this recent work we will try to answer the following questions using a latticed square with \( \text{pixels} \times \text{pixels} \) lattice points:

1. Is it possible to find the points that belong to the tessellation in a simple way?

2. The classical approach to the tessellation assumes that the signals start from the various nuclei at the same time with equal propagation velocity. Is it possible to find an algorithm able to describe the situation when the signals starting from the various nuclei are randomly distributed in time and the propagation velocity is given by some physical rules?;

3. And finally, and more important in our case, could we model the SN explosions in the galactic plane and their interaction with the aid of the dynamical tessellation?;

In order to answer these questions we show in Sect. 2 how to construct the Voronoi figures using a latticed square. In Sect. 3 we develop a dynamical method able to describe the intermediate phases and Sect. 4 is devoted to the astrophysical applications.

2. The lattice algorithm

A classical definition of the Voronoi cell could be found in Icke and Van de Weygaert (1987): “The points of a Voronoi cell all share the property that they are closer to the cell’s nucleus than any other”.

Here we should always associate the word points with the word pixel. In the following we will introduce some variables which have the same meaning in the algorithm used and in the plots and in the text.

First of all we pick a random distribution of nuclei.

We now consider a lattice \((\text{pixel})^2\) and associate to every point a little area \(l \times l\) where \(l = \text{side/pixel}\). The maximum value of \(\text{pixel}\) depends on the computer’s dynamical memory (in our case could be 500 or less) and \(\text{side}\) is the length of the square. In order to find the lattice’s points (little square) that belong to the Voronoi’s cells we use the following two step algorithm:

1. At the center of every pixel we compute the distances from the nuclei. We then sort the obtained vector into ascending order; we then select the possible candidates of the lattice’s points giving the condition that the difference between the first two elements of the sorted array is less than \(\sqrt{2}l\).

2. Once the first class of candidates is obtained we draw a line between the two nearest nuclei. In the mid point we draw a perpendicular line (perpendicular bisector) and we check if intersect or not the little square associated with the considered pixel.

The nodes are then computed with a similar algorithm but taking into account the three nearest points rather than two. A typical run is shown in Fig. 1, where \(\text{num}\) is the number of the little squares derived. A second run is performed, inserting a gaussian distribution of points around the center with standard deviation \(\sigma\), see Fig. 2.

3. Dynamical Voronoi tessellation

The classical approach to the tessellation assumes that all the injection points start at the same time and the velocity of
propagation is equal, or that at the selected time the expanding holes have all collided.

Conversely we will now explore the situation in which the injection points are randomly distributed in time (and space) and we will look at time-evolving situations rather than static ones: these configurations are named Johnson-Mehl tessellation, see Johnson and Mehli (1939) and Meyering (1953). The walls should now become hyperbolic sections rather than straight lines.

First of all, due to the enormous complexity of the problem an unavoidable restrictive hypothesis is made that will always be assumed in the following: the injection points should be selected in order that, at the given time, no expanding shell has yet reached the surrounding nuclei.

We therefore generate a temporal random distribution of nuclei each one labeled with the time at which the process has started:

\[ t_i = R [0, \text{time}] , \]

where \( R \) indicates a random number. We therefore assume that the shells expand at constant velocity \( (V_{\text{exp}}) \). The radius of each one satisfies:

\[ \text{Radius}_i = (\text{time} - t_i) \times V_{\text{exp}} . \]

Applying the lattice algorithm we analyze in each little square if there is intersection or not with the circles drawn from the various sources. A typical situation is represented in Fig. 3.

3.1. Random nuclei distribution

In order to cover a wide range of parameters we choose the time over which we allowed the explosions on the basis of the following
Fig. 4. The dynamical Voronoi tessellation. The collided little squares and those still expanding are drawn through little black squares. The nodes are marked.

Fig. 5. Dynamical Voronoi tessellation but with a gaussian distribution of nuclei. The nodes are marked.

Fig. 6. Dynamical tessellation with a random spatial/temporal distribution of SN in the galactic plane. The nodes are marked.

Fig. 7a. Emissivity contours from the tessellation.
3.2. Gaussian nuclei distribution

Taking a gaussian distribution the same two steps process could be applied using the fundamental restrictive hypothesis, see Fig. 5. Now the time is fixed a priori.

4. Astrophysical applications

The theory already developed can be easily transferred to study the SN explosions and their eventual interaction in the galactic plane.

The parameters of the spiral galaxy and the considered area should therefore be introduced:

\[
\begin{align*}
SN \text{ rate} & = \text{number of explosions every } 100 \text{ yr}, \\
\text{time} & = (\text{considered time})/10^6 \text{ yr}, \\
AC & = s\text{ide}^2 \text{ (pc), considered area}, \\
AI & = R\text{adius}^2 \text{ (pc)} \times \pi \times \text{filling, area interested in the explosions}, \\
\text{filling} & = \text{stars area}/\text{total circle area}.
\end{align*}
\]

Fig. 7b. The same like Fig. 7a but with a color coding (8 colors) and 300 x 300 pixels
The number of events will become:
\[ n_{\text{nuclei}} = \text{SN rate} \times \text{time} \times \frac{AC}{AI}. \]  
(5)

Once we know how many SNs are exploding in the considered area we should find a velocity behavior for the expanding shells.

The supernovae explosions could be characterized by 3 phases, see Dalgarno and Layzer, 1987 and Rohlfs 1986:

1. A first one with \( v_{\text{ex}} \approx 10^5 \text{km s}^{-1} \) up to \( 200(M_{\odot}/M_{\odot})^{1/3} n_0^{1/3} \) yr, were the interstellar medium density is \( \rho_0 = n_0 \cdot m_e \cdot m_n \) the number of particles/cm\(^3\) and \( M_{\odot} \) the ejected mass. This ends when the ejecta encounter a comparable mass of ambient interstellar gas.

2. We then have a blast wave (the shell) that is expanding hypersonically. The radius increases with the law:
\[ R(t) = \left[ \frac{25 E_0}{4 \pi \rho_0} \right]^{1/5} t_4^{2/5} \text{ pc}, \]
(6)
where \( E_\odot = E_0/10^{51} \text{ erg} \) and \( t_4 = t/10^4 \text{ yr} \). This energy conserving phase ends when \( \int L(t) \, dt \approx 0.3 E_0 \) with \( E_0 \approx 10^{51} \text{ erg} \) and \( L(t) \) the instantaneous power radiated from the shell; that time is \( t_\text{rad} = 3 \times 10^9 E_\odot^{1/2} \rho_0^{-0.55} \) yr and \( R_\text{rad} = R(t_\text{rad}) \).

3. We finally have a phase in which there are no longer any pressure forces to drive the shock. The shell will move at a constant radial momentum pilling the swept-up interstellar gas:
\[ R(t) = R_\text{rad} \left( \frac{8 t}{5 t_\text{rad}} - \frac{3}{5} \right)^{1/4}. \]
(7)

\[ \begin{align*}
\text{Dynamical Voronoi foams} & \quad n_{\text{nuclei}}=9 & \quad \text{Emissivity Surface pixels}=360 \\
\text{time [ML]} = 0.70 & \quad \text{side[pc]}=200.00 & \\
\text{Rate[SN/100y]} = 0.01 & \quad \text{E[10**61 ergs]}=1.00 \\
\text{R[pc]} = 10000.00 & \quad \text{AC[pc*pc]} = 400000.00 \\
\text{Flux} = 1000.00 & \quad \text{Filling}=0.10 \\
\phi=30.00 & \quad \theta=60.00 & \quad \text{distance}=100.00 \\
\text{zincr}=100.00 & \\
\end{align*} \]

Fig. 7c. Isometric surface of emission

Due to the large value of the involved lengths we take the expansion law of phase 2 and 3. Of course we should discriminate between the two phases \( t > R_\text{rad} \) and \( t < R_\text{rad} \).

At this point the problem becomes nonlinear but the same general two steps algorithm (once the subroutines that give the distances/the times are changed) could be applied, see Fig. 6. Here the main assumption is the constant density of the surrounding medium. We remember that the ambient interstellar gas may be modified by the pre-supernova stars and stellar winds. In this dynamical situation the gas could be of fairly uniform density.

4.1. The emissivity contours

In the previous paragraphs we have outlined a possible scenario for the SN explosions and their consequent interaction. We should not forget that in effects we observe synchrotron radiation or optical shocked emission.

A possible way to simulate the emissivity contours is to associate with a certain number of the little squares derived a flux \( \phi \) in arbitrary units with the following tentative scaling rule:

(i) \( 1 \times \text{flux} \phi \) for the small squares still expanding,
(ii) \( 2 \times \text{flux} \phi \) for the already collided little squares,
(iii) \( 3 \times \text{flux} \phi \) for the nodes.

The spatial distribution at the distance \( \theta \) will be of the type:
\[ \phi(\theta) = \phi(0) e^{-0.5 \theta^2}, \]
where \( \theta \) is the distance after which the value of the flux is 1/e the central one. Given these recipes we then scan a 100 \times 100 grid taking in each point the maximum flux from the various sources. A typical example is shown in Fig. 7a using the contour of isoflux with a standard algorithm, in Fig. 7b with an 8-color coding and in Fig. 7c using the isometric surface. Using the isometric surface the distance of the observer (in workbox units) the vertical angle \( \theta \) (in degrees) from the \( x-y \) plane and the angle in the \( x-y \) plane \( \phi \) (in degrees) should be introduced.

5. Conclusions

The local analysis on the little squares of a \( \text{pixels} \times \text{pixels} \) squared lattice seems powerful and able to describe both the stationary and the dynamical situation. The computing time (cpu seconds on the VAX 8600/32 Mb) is 12 seconds per nuclei in the classical tessellation and 36 seconds per nuclei in the dynamical case.

The dynamical analysis could be tentatively applied to describe the explosions and the consequent interaction of the SN in the galactic plane. The zones of temporal tessellation represent a moderate increase in the local density and a possible plane partition. The nodes with the highest density enhancements constitute a natural candidate to trigger the star formation.

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