Dynamical Voronoi tessellation

II. The three-dimensional case

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Abstract. For a given configuration of nuclei in a three-dimensional space a procedure for constructing the corresponding Voronoi diagram is presented. We then derive a similar procedure to explore the case in which the signals starting from the nuclei are still expanding. We tentatively apply the theory to the SN explosions in the galactic center and to the cluster's arcs.

Key words: interstellar medium: bubbles – interstellar medium: radiation field – supernovae and supernova remnants: general – clusters: of galaxies

1. Introduction

The field of the Voronoi chains presents a renewed interest due to the possible cosmological applications concerning both the galaxy distribution (see Yoshioka and Ikeuki, 1989) and the cluster distribution (see Icke and Van de Weygaert, 1987, 1989).

The problem could be split in two classes: the first one comprises the various algorithms able to reproduce the walls, the edges and the nodes that the 3-D distribution of nuclei produce, and the second one includes the astrophysical applications that range from the formation of clusters, extragalactic arcs, galaxies to SN explosions and subsequent formation of filaments and stars.

Here we extend an algorithm previously derived in two dimensions, see Zaninetti (1989, hereafter Paper I) to three dimensions.

We start with a lattice by \((pixels)^3\) points filling a given portion of the space and we answer the following questions:

1) Can we develop a slice method able to describe the intersection of a plane with the faces of the irregular polyhedra?
2) Can we model the appearance of nodes, edges and faces that comes out from the dynamical expansions?
3) Is it possible to simulate the intersections of SN explosions in an active astrophysical space?
4) Are the luminous arcs in cluster of galaxies the result of great explosions?

2. The scanning algorithm

Recently the 3-D problem has been solved using a geometrical model coupled with a kinematic process, see Icke and Van de Weygaert (1989).

In the following we will work on a three-dimensional lattice defined by \(pixels \times pixels \times pixels\) points, \(L_{knn}\).

Consider a set of centers \(i\) located at position \(x_i\). The Voronoi polyhedron \(V_i\) around a given center \(i\), is the set of lattice points \(L_{knn}\) closer to \(i\) than to any \(j\): more formally,

\[
L_{knn} \in V_i \iff |x_{knn} - x_i| \leq |x_{knn} - x_j|,
\]

where \(x_{knn}\) denotes the lattice point position. Thus, the polyhedra are intersections of half-spaces. Given a center \(i\) and its neighbor \(j\), the line \(ij\) is cut perpendicularly at its midpoint \(y_{ij}\) by the plane \(H_{ij}\).

We call \(H_{ij}\) the half-space generated by the plane \(H_{ij}\), that consists of the subset of lattice points on the same side of \(H_{ij}\) as \(i\); therefore

\[
V_i = \cap_{j} H_{ij},
\]

Fig. 1. The random distribution of nuclei in the space
Fig. 2a–c. The intersection of the faces of the polyhedra with a two-dimensional lattice; plane, varying from 1 to pixels, gives the z-position

2.1. The thick slice method

First of all we pick a distribution of nuclei, that in our case will be Gaussian or random. Given a section of the cube of a certain thickness $l = \text{side}/\text{pixels}$ the half planes forming the various $V_i$ may
or may not cross the little cubes belonging to the two-dimensional lattice. The following two-step algorithm has been used:

i) At the center of every lattice points $L_{kmn}$, $k$ fixed $m$ and $n$ varying, we compute the distances from the nuclei. We sort the obtained vector into ascending order and we then select the possible candidates giving the condition that the difference between the first two elements of the sorted array is less than $\sqrt[3]{l}$.

ii) Once the first class of candidates is obtained we draw a line between the two nearest nuclei. At the mid-point we draw a perpendicular plane and we check if it intersects or not the little cube associated with the considered point.

A typical run is shown in Fig. 2a–c where a random distribution of points is considered (see Fig. 1). The thickness of certain lines is due to the low angles between faces and scanned plane. Of course now the projection effects are important and the latitude, the longitude and the distance in a work-box units of the point of view of the observer should be introduced.

A second run is performed, inserting a Gaussian distribution of points around the center with standard deviation $\sigma$ as in Fig. 3, see Fig. 4a–c.

This scanning requires about 10 seconds/nuclei of CPU time on the VAX 8600/32 Mb.

2.2. The cubic lattice

Given $(\text{pixels})^3$ little cubes associated to $L_{kmn}$ the following definitions should now be introduced:

faces $\Rightarrow$ an element $L_{kmn}$ belongs to a face of the Voronoi polyhedra when the difference of the distances from the two nearest nuclei is less than $l/\sqrt{3}$ and it is intersected by the two bisector planes.

edges $\Rightarrow$ an element $L_{kmn}$ belongs to an edge of the Voronoi polyhedra when the difference of the distances from the three nearest nuclei is less than $l/\sqrt{3}$ and it is intersected by the three bisector planes.

nodes $\Rightarrow$ an element $L_{kmn}$ belongs to a node of the Voronoi polyhedra when the difference of the distances from the four nearest nuclei is less than $l/\sqrt{3}$ and it is intersected by the four bisector planes.

Given these recipes we then scan the volume and extract the points that belong to the edges and to the nodes. A typical run is shown in Figs. 5 and 6. The reported configuration requires 180 s/nuclei of CPU time on the VAX 8600/32 Mb.
Fig. 5a and b. The displacement of nodes and edges in the space and the relative random distribution of nuclei.

Fig. 6a and b. The same as in the previous figure but with a Gaussian distribution of points.
3. The dynamical case

When the signals starting from the various nuclei are randomly distributed in time and their velocity follows a physical rule it becomes interesting to look at the intermediate situations.

As in Paper I we choose configurations in which the expanding spheres have not yet reached the surrounding nuclei (we consider this a fundamental hypothesis).

We start by generating a temporal distribution of nuclei:

\[ t_j = R[0, \text{realtime}], \]  
\[ \text{where realtime is the time at which we freeze the situation. We therefore assume a given dependence with time:} \]

\[ \text{Radius} = F(\text{time} - t_j). \]  

The function \( F \) is specified from the physical context and we start with a constant velocity equal to the unit.

We are now generating polyhedra that are intersections of hyperboloid rather than planes. In order to continue we should introduce the concept of time-distance. Given a lattice element \( L_{x,m} \) the time-distance is the time that the signals starting from the various nuclei take to arrive at the considered element plus the time delay due to the random temporal distribution. In each pixel we therefore reorder the various times in ascending order. The smallest time is associated with the first signal to arrive.

The previous geometrical definitions should now be modified: faces \( \Rightarrow \) an element \( L_{x,m} \) belongs to a face of the dynamical polyhedra when the temporal distances from two nuclei are greater than the real time and their difference is less than the time needed to cross the element,

edges \( \Rightarrow \) the same as above applied to three nuclei,

nodes \( \Rightarrow \) the same as above applied to four nuclei.

A typical run is shown in Fig. 7 in which the edges and the nodes are marked. Also the still expanding shells are shown through little points.

4. The galactic center

The high angular resolution radio observations of the galactic center present many radio emitting structures that assume a filamentary shape, see for example Bally and Yusef-Zadeh (1989). These could be the fossil records of ancient explosions and we are now going to explore this hypothesis.

We remember that in spiral galaxies the surface brightness of the disk is an exponential function of the radius:

\[ S(r) = S_0 \exp\left(-\frac{r}{r_d}\right), \]  

with the disk scale length \( r_d = 3.5 \) kpc.

Along the disk the density probability to have a SN explosion will scale like:

\[ \rho(r) = \frac{1}{r_d} \exp\left(-\frac{r}{r_d}\right), \]  

and therefore the probability to have a SN explosion with radius between 0 and \( r \) will be:

\[ 1 - \exp\left(-\frac{r}{r_d}\right). \]
Given the time, \( t_0 \) \( 10^6 \) yr, and the SNRATE = explosions/yr, the number of SN events is given by:

\[
SN = t_0 \times 10^6 \times \text{SNRATE} \left[ 1 - \exp \left( -\frac{1}{r_d} \right) \right].
\]

This is also the number that should be introduced in our computations, see Fig. 8. The results, inserting the radius of the SN expansion as given in Paper I, are reported in Fig. 9.

### 4.1. The emissivity contours

As in Paper I a possible way to simulate the emissivity contours is to associate maximum emitting fluxes with the derived entities. The following tentative scaling rule has been used:

(i) \( 1 \times \text{flux} \) for some little cubes still expanding,
(ii) \( 2 \times \text{flux} \) for some little cubes belonging to the edges,
(iii) \( 3 \times \text{flux} \) for the nodes.

The spatial distribution at the distance \( \rho \) will be of the type:

\[
\text{flux} (\rho) = \text{flux}_0 e^{-\rho^2 / \rho^2},
\]

where \( \rho \) is the distance after which the value of the flux is \( \approx \) half the central one. Given these recipes we then scan a \( 100 \times 100 \) grid in the \( y-z \) plane taking in each point the maximum of the 100 maximum fluxes (from the various sources) along \( x \). A typical example is shown in Fig. 10a in which the isoflux contours are shown and in Fig. 10b with an eight color coding and where a \( 300 \times 300 \times 300 \) grid has been used.

### 5. Extragalactic arcs

Recently the "extragalactic zoo" has been enlarged by the discovery of the luminous arcs in clusters of galaxies, see Lynds and Petrosian (1989). They have the following properties: location in clusters of galaxies, narrow arclike shape, enormous apparent length and center of curvature directed towards the center of gravity of the cluster. Many models have already been proposed and now we suggest the intersection of the expanding shells. The radius of an explosion can be estimated using the Sedov solution, Sedov (1959), which describes the selfsimilar expansion of the shock during the adiabatic phase:

\[
R = 1.1 \left[ \frac{E}{\rho} \right]^{1/5} t^{2/5}.
\]

Inserting physical values appropriate for the cluster environment we obtain:

\[
R = 203 \left[ \frac{E_{61}}{N_{-5}} \right]^{1/5} t^{2/5} \text{kpc}.
\]

Choosing a cube of side \( = 500 \) kpc, expressing the time in \( 10^7 \) yr units, the energy in \( 10^{61} \) ergs units, inserting the first explosion at the center followed by few other events, we obtain the curved edges that are shown in Fig. 11.

Also, with a procedure similar to that of the previous paragraph, we can obtain simulated contour brightness of shocked emission, see Fig. 12.
6. Conclusions

The intersecting shells scenario could be a powerful tool to explain a wide variety of astrophysical phenomena ranging from the SN explosions to cosmological detonations. The problem, rather complicated due to the random distribution of seeds both in time and space, could be solved with the help of the Voronoi diagrams.

Exploring the time delay from the various nuclei we can construct the quasi-polyhedra defining the edges by the intersection of three shells and the nodes by the intersection of four. The situation is dynamical and we explore intermediate situations.

The resulting curved edges coupled with a three-dimensional algorithm allow to simulate the contours of radio/optical brightness and to explain complicated morphologies like the filamentary radio emission at the galactic center or the luminous arcs in clusters.

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Luminous Arc in Clusters
points = 5  a = 500.00[kpc]  pixels = 400
longitude = 30.00  latitude = 45.00  distance = 5000.00
nodes = 0  time = 2.00[10^-7 years]
L = 1.00 * 10**6[ergs/sec]  n5 = 1.00[10**-5 part/cm**3]

Luminous Arc - Radioemission
points = 5  a = 500.00[kpc]  pixels = 400
nodes = 0  time = 2.00[10^-7 years]
L = 1.00 * 10**6[ergs/sec]  n5 = 1.00[10**-5 part/cm**3]
flux0 = 1000.00  a = 8.00[kpc]

Fig. 11. The dynamics of the cluster's explosions. The nodes, the edges, the not yet collided shells are reported with symbols of respectively decreasing size.

Fig. 12. The simulated contour emissivity from the arc.

References