Dynamical Voronoi tessellation

IV. The distribution of the asteroids

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Abstract. We analyze the volume distribution of asteroids with the aid of lognormal and fractal distribution. We therefore compute volumes through Voronoi diagrams and we try to simulate the observed distribution using a fragmentation process to generate the seeds.

Key words: minor planets – methods: statistical

1. Introduction

The “exploded planet” hypothesis for the origin of the asteroids is nearly as old as the first discovery of an asteroid in 1801.

But this theory has been abandoned due to two difficulties:

(i) how to produce the observed distribution of asteroid taxonomic types having their distinct ordering with semimajor axes (Gradie & Tedesco 1982);

(ii) the problem of identifying a plausible disruption mechanism and the fact that many asteroids are in orbits that do not and probably never did overlap each other, and it is hard to understand how these bodies could be fragments from the same parental bodies.

Despite these difficulties, we went through the “fractal” nature of the sizes of the asteroids and we delineate a possible theoretical mechanism of fragmentation through Voronoi diagrams with a particular choice of the seeds.

Given these starting points we shifted from the “naive” idea of fragmentation of a unique body to:

(a) the fragmentation of many solid “objects” due to an internal mechanism,

(b) the fragmentation of solid bodies due to catastrophic collisions between asteroids.

Our sample of bodies belonging to the asteroids range from a few km of the minimum diameter to 913 km of the maximum one and has been extracted from the IRAS data base of asteroids albedos (for practical purpose we refer to the data analyzed by Zappalà et al. 1990). Here in order to have the basis of further developments we fit the experimental distribution of the data through a lognormal distribution, see paragraph 2, and a fractal one but this time selecting only the upper part of the observed data.

We then go on to explore the possibility that the asteroids originate from the explosion of a unique body when the broken faces follow those of the Voronoi polyhedra, see paragraph 3.

2. The distribution of the asteroids

Given \( N = 4100 \) data of diameters, we compute the frequency distribution using classes with the boundaries spaced regularly in the logarithm of the diameters (base 10), and assuming that the values in the interval are concentrated at the midpoint. The obtained final distribution is reported at the top of Fig. 1.

We then try to fit the data with two different distributions over different ranges.

2.1. The lognormal distribution

A very useful model is the well known Gaussian distribution given by:

\[
 f(x) = \frac{1}{\sqrt{2\pi \sigma}} \exp\left(-\frac{(x-x_\text{avg})^2}{2\sigma^2}\right), \tag{1}
\]

where \( x \) is the asteroid diameter, \( x_\text{avg} \) the averaged diameter and \( \sigma \) the standard deviation. However, due to the nature of the problem we are mainly interested in a logarithmic variant:

\[
 f_{\ln}(x) = \frac{1}{\sqrt{2\pi \ln \sigma}} \exp\left(-\frac{(\ln x - \ln \bar{x})^2}{2\ln^2 \sigma}\right). \tag{2}
\]

The parameters \( \bar{x} \) and \( \sigma \), which characterize the distribution denote the statistical median and the standard deviation as defined by:

\[
 \ln \bar{x} = \frac{\sum n_i \ln x_i}{\sum n_i}, \tag{3}
\]

\[
 \ln \sigma = \left[\frac{\sum n_i (\ln x_i - \ln \bar{x})^2}{\sum n_i}\right]^{1/2}. \tag{3}
\]
where \( n_i \) is the number fraction of particles at an interval of the size histogram centered around \( x_i \).

As used herein \( \ln \) specifies logarithms to the base \( e \), and \( \log \) will be reserved for the base 10.

If now we want to speak of a theoretical distribution we can follow the arguments of Granqvist & Buhrman (1976) and the theoretical frequency distribution of \( x \) is therefore given by:

\[
n_i = N \frac{\Delta x_i}{x} f_{LN},
\]

where \( \Delta x_i \) is the magnitude of the size interval and due to the logarithmic behavior is different for each bin.

It is straightforward to show that the fraction \( F_{LN}(x) \) of the total number of asteroids which have a diameter smaller that a certain size \( x \) is given in terms of the error function:

\[
F_{LN}(x) = \frac{1}{2} + \frac{1}{2} \text{erf} \left( \frac{\ln(x/x)}{2\sqrt{2} \ln \sigma} \right).
\]

This equation points directly toward the usefulness of log-log plots on which \( F_{LN}(x) \) versus \( x \) yields a straight line.

We report in Fig. 2 the diameter distribution. It should be noted that the cumulative percentage has been plotted against the upper limit of the size histogram; this is the only way of transforming the histogram into a probability plot.

At the end of this section it is important to stress the high value of the standard deviation \( \ln \sigma = 0.93 \), this is due to the fact that the diameters considered span over 3 decades.

\[2.2. \text{ The fractal distribution}\]

In order to continue we organize the data in a way similar to that adopted in previous studies on fragmentation. In such studies the number of fragments \( N \) with a characteristic linear dimension greater than \( r = V^{1/3} \) are usually plotted as a function of \( r \). In our case we make the same numerical treatment and the results are shown in Fig. 3.

The basic data set is the size-frequency distribution of asteroids as published by Cellino et al. (1991); this distribution is known to be incomplete at sizes smaller than \( \approx 44 \) km and the degree of incompleteness becomes larger with smaller-sized asteroids. Thus the number of asteroids and their associated volumes will decrease with decreasing size and will go to zero below the minimum diameter (\( \approx 1 \) km). This of course, does not mean that there are no
asteroids smaller than 1 km in fact the best estimates are that there are millions of sub-km diameter asteroids.

In order to compute the true distribution of asteroids volume, one uses a bias-corrected size distribution of asteroids or else limits the analysis to sizes above the completeness limit. In order to find a scientific way to produce such a reduction we make a linear fit of the logarithm of the involved quantities:

$$\log(\text{frequency}) = a + b \log(\text{middle class value}).$$  \hspace{1cm} (6)

In this way we can limit ourselves to the values that gives a linear correlation coefficient in absolute value greater than 0.99; adopting this stance the diameter distribution now becomes a fractal with the dimension given by the coefficient \( b \). The lower limit of the distribution is \( \approx 40 \text{ km} \), see Fig. 3b, and the fractal dimension \( D = 2.5 \pm 0.20 \), see Fig. 4. This value is in perfect agreement with a fractal dimension for a variety of fragmented objects such as broken coal, interstellar grains, projectile fragmentation of quarsite, and basalts, see Turcotte (1989). We should remember that the values of the fractal dimension of various classes of fragmented objects vary considerably but most lie in the range \( 2 < D < 3 \).

3. The simulation

If a given material undergoes a mechanical fragmentation its size is distributed with a certain probability: its exact

![Fig. 3a](image1)

**Fig. 3a.** The logarithm of the number of asteroids with a characteristic linear dimension greater than diameter as a function of the logarithm of the diameter

![Fig. 3b](image2)

**Fig. 3b.** The logarithm of the number of asteroids with a characteristic linear dimension greater than diameter as a function of the logarithm of the diameter when a lower cut is introduced

![Fig. 4](image3)

**Fig. 4.** The same as Fig. 3b but with more bins and the evaluation of the fractal dimension. The interpolating line is represented with a thick line
determination is still an argument of research. The first theory was developed by Sir Neville Mott (1947): his theory assumes a time dependent probability of fracture of a ring after a critical strain has been reached in the material. On this basis the form of the particle size distribution can be determined empirically. When the resulting distribution is analyzed, the frequency of occurrence of particular masses follows a cumulative distribution of the form:

$$\frac{N_m}{N} = e^{-\frac{m}{\mu}}$$  (7)

where $N_m$ is the number of fragments, each of them having a mass greater or equal to $m$, $N$ is the total number of fragments,

$$\mu = \frac{\bar{m}}{2},$$  (8)

and $\bar{m}$ is the averaged mass of a fragment.

Another approach is through the Voronoi Polyhedra. Assuming that a material breaks through preexisting planes of weakness that follow faces of equal distance between two different seeds we compute the volumes associated with each seed.

In a first simulation Kiang (1966) working on a lattice of $20^3$ points and 100 seeds randomly distributed fitted the normalized distribution with the curve:

$$H(x; c) = \frac{c}{\Gamma(c)} x^{c-1} e^{-cx},$$  (9)

with $c = 6$; starting from this milestone we can continue increasing the pixels of the lattice and changing the rules of seeds production and data fit.

3.1. Fractal nuclei

If the number of objects $N$ with a characteristic linear dimension greater than $R$ satisfies the law:

$$N(R) \approx \frac{C}{R^D} \text{ for some positive constant } C,$$  (10)

a fractal distribution is defined. In order to produce nuclei that generate volumes resulting in a fractal distribution the following procedure should be adopted.

A cube with a linear dimension named “side” is our cell; this cell is divided into $(nn)^3$ little cubes; each with a dimension side/nn and called first order elements.

The probability that a cell will be fragmented into $(nn)^3$ elements is prob and we will therefore have $nint \times (\text{prob} \times (nn)^3)$ possible sites for further fragmentation.

We now insert into our list of nuclei the $(nn)^3 - nint \times (\text{prob} \times (nn)^3)$ non selected first order cells taking the coordinates of the central point of each one.

The second order cells have typical dimensions side/nn and now the procedure repeats itself; this process is stopped once a given number of nuclei is reached.

In first approximation the volume of the $n$th order cell $V_n$ is

$$V_n = \frac{V_0}{nn^{3n}},$$  (11)

where $V_0$ is the volume of the zero order cell. After fragmentation the number of zero order cells $N_0$ is

$$N_0 = (1 - \text{prob}) N_0,$$  (12)

where $N_0$ is the number of zero order before fragmentation. After fragmentation the number $n$th order cells $N_n$ is

$$N_n = (nn^3 \times \text{prob})^n (1 - \text{prob}) N_0.$$  (13)

Combining the equations we obtain:

$$\ln \frac{V_n}{V_0} = -n \ln(nn^3),$$  (14)

$$\ln \frac{N_n}{N_0} = -n \ln(nn \times \text{prob})^3.$$  (15)

Eliminating $n$ we obtain:

$$\frac{N_n}{N_0} = \left[ \frac{V_n}{V_0} \right]^{-\ln(nn^3 \text{prob}) / \ln(nn^3)}.$$  (16)

This is a fractal distribution with dimension given by

$$D = \frac{3 \ln[(nn^3 \times \text{prob})]}{\ln(nn^3)}.$$  (17)

In effect for numerical purposes

$$D = \frac{3 \ln[nint(nn^3 \times \text{prob})]}{\ln(nn^3)},$$  (18)

where we have used the symbol nint to indicate the nearest integer to the argument between parentheses. In the following section we will go on to compute the volumes associated with each nucleus using the Voronoi diagrams that will reflect these theoretical arguments.

3.2. Our algorithm

We work on a 3D lattice $L_{k,m,n}$ of $200^3$ elements and the following recipes should be adopted:

(a) we start with a big boundary volume in which we insert the seeds,

(b) to take account of the boundary conditions we considered a smaller volume in which we placed our lattice,

(c) the exact ratios between the two volumes are specified in the plots.

We should remember that a Voronoi polyhedron is the set of lattice points nearer to a given nucleus than to any other nucleus. But it turns out that the distribution in volume of such polyhedra is extremely sensitive to the seed distribution, see Zaninetti (1992), and we limit ourselves to the fractal points, see Fig. 5a for what concerns the seeds.
distribution in space and Fig. 5b concerning the derived volume distribution.

In this case due to the low resolution (250 x 250 x 250 pixl) the volumes are computed with a low accuracy and we obtain a fractal dimension (the coefficient b of the least square fit) $D = -3.66 \pm 0.32$ against the theoretical one of 2.32.

We are therefore obliged to go in the direction of a greater resolution.

3.3. An approximate fast algorithm

In order to produce a fast computation of the volumes and to reach a great resolution (5000 x 5000 x 5000 pixl) the following approximate method has been developed.

We start with the same seeds as in the previous paragraph but we consider only a certain number of volumes associated with seeds with progressively increasing distances from the center of the box.

From each seed we draw a line along the x-axis adding each time a distance:

$$\delta = \frac{\text{side}}{\text{pixels}}$$

(19)

Our radius will be defined when we change the influence of the nearest seed.

Fig. 5a. The nuclei represented with spheres at the end of little vertical sticks

Line simulation using fractal with prob= 0.63 and nn=2
selected box nuclei=149 box nuclei=400 pixels=5000
side=200.00 Volume great box= 2.00
a = 9.59 sa= 0.13 b=-2.68 sb= 0.41 r=-0.98
b-theoretical =-2.32

Fig. 5b. The logarithm of the number of Voronoi polyhedra with a characteristic linear dimension greater than diameter using fractal seeds

Fig. 6. The logarithm of the number of Voronoi polyhedra with a characteristic linear dimension greater than diameter using fractal seeds when a fast algorithm to compute the volumes is used.
4. Conclusions

The new contributions of this paper can be considered from data fit and astronomical points of view:

(i) The volume distribution of the asteroids could be fitted through a unique probability distribution throughout the existence range: the lognormal, i.e. $f_{LN}$.

(ii) The diameter distribution is known to be incomplete at sizes smaller than $\approx 30$ km, and we therefore fit the upper part of the diameter data through a fractal distribution. The obtained dimension turns out to be 2.5, a value that is in agreement with the measure obtained from various types of experiments on fragmentation.

Obtaining a theoretical model able to produce a scale invariant distribution of the linear dimensions is more problematic. We have reached this objective by using a fragmentation algorithm to produce the seeds. Computations of the volumes/diameters through a fast method give scale invariant results with a dimension which is slightly different from what results given the assumption of cubic volumes associated with each seed.

From an astronomical point of view there are two prospects for further work:

(i) to apply the concept of fractal behavior to a more restricted samples, such as the families,

(ii) to explore the behavior of surveys of smaller size asteroids to confirm such possible “fractal” behavior.

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References

Gradie J., Tedesco E.F., 1982, Sci. 216, 1405
Kiang T., 1966, Z. Astrophys. 64, 433